

SOLUTIONS OF BIGRADED TODA HIERARCHY

CHUANZHONG LI

Department of Mathematics, USTC, Hefei, 230026, Anhui, P. R. China
Department of Mathematics, NBU, Ningbo, 315211, Zhejiang, P. R. China

ABSTRACT. The (N, M) -bigraded Toda hierarchy is an extension of the original Toda lattice hierarchy. The pair of numbers (N, M) represents the band structure of the Lax matrix which has N upper and M lower diagonals, and the original one is referred to as the $(1, 1)$ -bigraded Toda hierarchy. Because of this band structure, one can introduce $M + N - 1$ commuting flows which give a parametrization of a small phase space for a topological field theory.

In this paper, we first show that there exists a natural symmetry between the (N, M) - and (M, N) -bigraded Toda hierarchies. We then derive the Hirota bilinear form for those commuting flows, which consists of two-dimensional Toda hierarchy, the discrete KP hierarchy and its Bäcklund transformations. We also discuss the solution structure of the (N, M) -bigraded Toda equation in terms of the moment matrix defined via the wave operators associated with the Lax operator, and construct some of the explicit solutions. In particular, we give the rational solutions which are expressed by the products of the Schur polynomials corresponding to non-rectangular Young diagrams.

Mathematics Subject Classifications (2000). 37K05, 37K10, 37K20.

CONTENTS

1. Introduction	2
2. Tridigonal Toda lattice hierarchy and generalization	3
3. The bigraded Toda hierarchy (BTH)	5
3.1. Example of the $(2, 2)$ -BTH	8
4. Equivalence between (N, M) -BTH and (M, N) -BTH	10
4.1. Equivalence in the Hirota bilinear identities	10
4.2. Equivalence in the Hirota bilinear equations	11
4.3. Equivalence in the Lax equations	12
4.4. Equivalence between $(1, 2)$ -BTH and $(2, 1)$ -BTH	14
5. Hirota equations and solutions of the BTH	15
5.1. Solutions of the BTH in the semi-infinite matrix	18
6. Rational solutions of the (N, M) -BTH	23
6.1. Rational solutions of the $(1, M)$ -BTH	27
7. Conclusions and discussions	28
References	29

Correspondence: czli@mail.ustc.edu.cn.

1. INTRODUCTION

The (N, M) -bigraded Toda hierarchy, denoted by (N, M) -BTH, is an integrable system (see e.g. [1, 2]), and its Lax operator is given by

$$\mathcal{L} := \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_0 + \cdots + u_{-M}\Lambda^{-M}.$$

where $N, M \geq 1$, Λ is a shift operator which can be expressed as an infinite matrix in the form, $\Lambda = (E_{i, i+1})_{i \in \mathbb{Z}}$. In terms of an infinite size matrix, the Lax operator \mathcal{L} has the band structure with N upper and M lower nonzero diagonals. The (N, M) -BTH is then defined by

$$(1.1) \quad \frac{\partial \mathcal{L}}{\partial t_{\gamma, n}} = \begin{cases} [(\mathcal{L}^{n+1-\frac{\alpha-1}{N}})_+, \mathcal{L}], & \text{if } \gamma = \alpha = 1, 2, \dots, N, \\ [-(\mathcal{L}^{n+1+\frac{\beta}{M}})_-, \mathcal{L}], & \text{if } \gamma = \beta = -M+1, \dots, -1, 0, \end{cases}$$

Here we call the $N + M - 1$ numbers of the flows for $n = 0$ the *primaries* of the BTH which describe the small phase space of a topological field theory (TFT), and the flows with $n > 0$ correspond to the gravitational descendants in this TFT.

In the case of $N = M = 1$, the $(1, 1)$ -BTH is the original Toda lattice hierarchy, and the primary flow is just the Toda lattice equation [3, 4]. The (N, M) -BTH with $N > 1$ or $M > 1$ is then considered as an extension of the original Toda lattice hierarchy. One should also note that the case with infinite N and M corresponds to the two-dimensional Toda hierarchy where we have two independent Lax operators (one defined near infinity and the other defined near zero in the spectral space, more precisely, one considers two cases with $(1, \infty)$ and $(\infty, 1)$). Then the (N, M) -BTH can be naturally considered as a reduction of the two-dimensional Toda hierarchy by imposing an algebraic relation to those two Lax operators (see [5–7]).

In [2], we showed the integrability of an extended version of (N, M) -BTH by writing the hierarchy as a bilinear identity, and introduced the τ -functions. Here the extension implies that the hierarchy has additional logarithmic flows, and this version is called the extended BTH (see [1, 8]). In this paper, we are interested in constructing several explicit solutions of the BTH, and as a first step, we only consider the non-extended version of the BTH based on our previous study [2].

The paper is organized as follows. In section 2, we give a brief summary of the original Toda lattice hierarchy whose Lax operator is given by a tri-diagonal matrix. We also discuss briefly $(2, 1)$ -BTH as an extension of the Toda hierarchy and describe the $t_{2,0}$ -flow defined by the square root of the Lax operator. In particular, we mention that there exists nonlocal terms in the equation, and make a remark that the flow also appears in the recent paper [9]. In section 3, we give the explicit form of (N, M) -BTH and the τ -functions. This section is a brief summary of our previous paper [2] without the logarithmic flows. Here we express the coefficient functions in the Lax operator in terms of the τ -functions in the similar manner discussed in [10, 13]. We also discuss some details of the $(2, 2)$ -BTH as an example. In section 4, we show the equivalence between the (N, M) - and (M, N) -BTHs by using the Hirota bilinear equations found in [2] and gauge transformation in [14]. To be illustrative, we also give some simplest concrete examples of equivalence between flows of $(2, 1)$ -BTH and $(1, 2)$ -BTH. This equivalence is explicitly shown in the examples given in section 5. In section 5, we derive the Hirota bilinear equations for the primaries of the (N, M) -BTH. Then we construct the τ -functions in terms of the moment matrix defined naturally via the wave operators introduced in section 3 (also see [2]). In section 6, we construct rational solutions based on the τ -function formulas derived in the previous section. In particular, those rational solutions are given by the products of two Schur polynomials depending on two different sets of flow parameters $t_{\alpha, n}$ and

$t_{\beta,n}$ in (1.1). Contrary to the case of the original Toda hierarchy where the rational solutions are given by the Schur polynomials of rectangular Young diagrams, the rational solutions of the BTH are parametrized by non-rectangular Young diagrams. Finally, in section 7, we summarize the results and give some discussions.

2. TRIDIGONAL TODA LATTICE HIERARCHY AND GENERALIZATION

Here we briefly explain the BTH as an extension of the original Toda equation, and present some connection to the recent study in [9]. The main point is to explain the structure of an additional symmetry generated by a fractional power of the Lax matrix. The Toda lattice equation is written in the form,

$$(2.2) \quad \begin{cases} \frac{\partial a_{n+1}}{\partial t_1} = a_{n+1}(b_{n+1} - b_n), \\ \frac{\partial b_n}{\partial t_1} = a_{n+1} - a_n. \end{cases} \quad n = 1, 2, \dots$$

Eq.(2.2) has the Lax representation with a tridiagonal semi-infinite matrix L by

$$(2.3) \quad \frac{\partial L}{\partial t_1} = [B_1, L], \quad B_1 = [L]_{\geq 0},$$

where $[L]_{\geq 0}$ is the upper triangular part of the matrix L given by

$$(2.4) \quad L = \begin{pmatrix} b_1 & 1 & 0 & 0 & \cdots \\ a_2 & b_2 & 1 & 0 & \cdots \\ 0 & a_3 & b_3 & 1 & \cdots \\ 0 & 0 & a_4 & b_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we consider the bounded Toda lattice equation, the Lax matrix will have finite size.

For semi-infinite Toda equation there exists a sequence of τ -functions $\{\tau_n : n \geq 0\}$ with $\tau_0 = 1$ defined by the a_n, b_n by the formulas

$$(2.5) \quad a_n = \frac{\tau_n \tau_{n-2}}{\tau_{n-1}^2}, \quad b_n = \frac{\partial}{\partial t_1} \log \left(\frac{\tau_n}{\tau_{n-1}} \right).$$

Then we can write the Toda lattice equation in the Hirota bilinear form,

$$(2.6) \quad D_1^2 \tau_n \cdot \tau_n = 2\tau_{n+1}\tau_{n-1},$$

where D_1 is the usual Hirota derivative. For the k -th flow-parameter t_k of the Toda lattice hierarchy, D_k is defined by

$$(2.7) \quad D_k f \cdot g := \left(\frac{\partial}{\partial t_k} - \frac{\partial}{\partial t'_k} \right) f(t_k) g(t'_k) \Big|_{t_k=t'_k}.$$

The hierarchy of the Toda lattice is defined by

$$(2.8) \quad \frac{\partial L}{\partial t_k} = [B_k, L], \quad B_k = [L^k]_{\geq 0}, \quad k = 1, 2, 3, \dots$$

The τ -functions of the Toda lattice hierarchy obey the following equations

$$(2.9) \quad [D_k - P_k(\hat{D})] \tau_{n+1} \cdot \tau_n = 0, \quad k = 2, 3, 4, \dots,$$

where Schur polynomial $P_k(\hat{D})$ is defined by

$$(2.10) \quad e^{\sum_{k=1}^{\infty} \frac{1}{k} D_k z^k} = \sum_{k=0}^{\infty} P_k(\hat{D}) z^k, \quad \hat{D} = (D_1, \frac{1}{2} D_2, \frac{1}{3} D_3, \frac{1}{4} D_4, \dots).$$

When $k = 2$, Hirota equation becomes

$$(2.11) \quad (D_2 - D_1^2) \tau_{n+1} \cdot \tau_n = 0.$$

Eq.(2.11) together with eq.(2.6) will give the nonlinear Schrodinger equation which can be seen as the second member of the Toda lattice hierarchy [11].

A natural question will be how about generalized band structure of the Toda Lattice hierarchy which is just $(1, 1)$ tridiagonal band matrix.

For example, for $(2, 1)$ Heissenberg band structure of Lax matrix

$$(2.12) \quad \tilde{L} = \begin{pmatrix} b_1 & c_1 & 1 & 0 & \cdots \\ a_2 & b_2 & c_2 & 1 & \cdots \\ 0 & a_3 & b_3 & c_3 & \cdots \\ 0 & 0 & a_4 & b_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

the equations for Toda flow, i.e.

$$(2.13) \quad \partial_{t_{1,0}} \tilde{L} = [\tilde{L}_{\geq 0}, \tilde{L}],$$

will lead to the following Blaszak-Marciniak lattice equation [14]

$$(2.14) \quad \begin{cases} \partial_{t_{1,0}} c_n &= a_{n+2} - a_n \\ \partial_{t_{1,0}} b_n &= c_n a_{n+1} - a_n c_{n-1} \\ \partial_{t_{1,0}} a_n &= a_n (b_n - b_{n-1}). \end{cases}$$

Eq.(2.14) is also equivalent to the Bogoyavlensky–Narita lattice given in [15]. In [14], Blaszak and Marciniak considered the local flows which correspond to the integer powers of Lax operators. In this paper, we construct nonlocal flows using fractional power of Lax operator. Because of the $(2, 1)$ -band structure, one can define the square root of Lax matrix which will be shown in detail in the next section. Using the operator $\tilde{L}^{\frac{1}{2}}$, we give a new flow

$$(2.15) \quad \partial_{t_{2,0}} \tilde{L} = [\tilde{L}_{\geq 0}^{\frac{1}{2}}, \tilde{L}],$$

which further leads to

$$(2.16) \quad \begin{cases} \partial_{t_{2,0}} c_n &= b_{n+1} - b_n + c_n (1 - \Lambda)(1 + \Lambda)^{-1} c_n \\ \partial_{t_{2,0}} b_n &= a_{n+1} - a_n \\ \partial_{t_{2,0}} a_n &= a_n (1 - \Lambda^{-1})(1 + \Lambda)^{-1} c_n. \end{cases}$$

After denoting \mathcal{H} as $\frac{1+\Lambda}{\Lambda-1}$, eq.(2.16) can be rewritten as

$$(2.17) \quad \begin{cases} \partial_{t_{2,0}} c_n &= b_{n+1} - b_n + c_n \mathcal{H}^{-1} c_n \\ \partial_{t_{2,0}} b_n &= a_{n+1} - a_n \\ \partial_{t_{2,0}} a_n &= a_n \mathcal{H}^{-1} c_n. \end{cases}$$

After transformation

$$(2.18) \quad c_n = \bar{c}_{n+1} + \bar{c}_n,$$

eq.(2.16) becomes

$$(2.19) \quad \begin{cases} \partial_{t_{2,0}} \bar{c}_{n+1} + \partial_{t_{2,0}} \bar{c}_n &= b_{n+1} - b_n + \bar{c}_n^2 - \bar{c}_{n+1}^2 \\ \partial_{t_{2,0}} b_n &= a_{n+1} - a_n \\ \partial_{t_{2,0}} a_n &= a_n(\bar{c}_n - \bar{c}_{n-1}), \end{cases}$$

which is just eq.(40) in [15] and also related to the system (10)-(12) proposed in [16]. In [15], Svinin gives general constructions of such flows. Studying the relation between two ways of constructing nonlocal flows, i.e. the way in [15] and the way in this paper might be an interesting problem.

Eq. (2.14) and eq. (2.17) are in fact t_1 flow and t_2 flow of the case $n = 3, \alpha = -1$ without center extension in [9], i.e. solutions do not depend on y variable. In this paper, we directly introduce a new fractional Lax matrix instead of using Casimir construction used in [9] for constructing the Lax equations. We also generalize these results to (N, M) -band matrix. For a finite-sized Lax matrix, its fractional power may not be well-defined. This leads to a difficulty to give symmetric flows generated by fraction powers of Lax matrix. But for a bi-infinite band matrix, one can define the fraction powers and further define other additional flows which commute with the original Toda flow. This generalization leads to the BTH which might be also seen as a general reduction of the two-dimensional Toda lattice hierarchy. In the next section, we give the continuous interpolated version of BTH [2] and later present the matrix version of the BTH in bi-infinite and semi-infinite band matrices.

3. THE BIGRADED TODA HIERARCHY (BTH)

The Lax operator of the BTH is given by the Laurent polynomial of Λ [1]

$$(3.20) \quad \mathcal{L} := \Lambda^N + u_{N-1}\Lambda^{N-1} + \dots + u_0 + \dots + u_{-M}\Lambda^{-M},$$

where $N, M \geq 1$, Λ represents the shift operator with $\Lambda := e^{\epsilon \partial_x}$ and “ ϵ ” is called the string coupling constant, i.e. for any function $f(x)$

$$\Lambda f(x) = f(x + \epsilon).$$

The \mathcal{L} can be written in two different ways by dressing the shift operator

$$(3.21) \quad \mathcal{L} = \mathcal{P}_L \Lambda^N \mathcal{P}_L^{-1} = \mathcal{P}_R \Lambda^{-M} \mathcal{P}_R^{-1},$$

where the dressing operators have the form,

$$(3.22) \quad \mathcal{P}_L = 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \dots,$$

$$(3.23) \quad \mathcal{P}_R = \tilde{w}_0 + \tilde{w}_1 \Lambda + \tilde{w}_2 \Lambda^2 + \dots$$

Eq.(3.21) are quite important because it gives the reduction condition from the two-dimensional Toda lattice hierarchy. The pair is unique up to multiplying \mathcal{P}_L and \mathcal{P}_R from the right by operators in the form $1 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \dots$ and $\tilde{a}_0 + \tilde{a}_1 \Lambda + \tilde{a}_2 \Lambda^2 + \dots$ respectively with coefficients independent of x . Given any difference operator $A = \sum_k A_k \Lambda^k$, the positive and negative projections are defined by $A_+ = \sum_{k \geq 0} A_k \Lambda^k$ and $A_- = \sum_{k < 0} A_k \Lambda^k$.

To write out explicitly the Lax equations of BTH, fractional powers $\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are defined by

$$\mathcal{L}^{\frac{1}{N}} = \Lambda + \sum_{k \leq 0} a_k \Lambda^k, \quad \mathcal{L}^{\frac{1}{M}} = \sum_{k \geq -1} b_k \Lambda^k,$$

with the relations

$$(\mathcal{L}^{\frac{1}{N}})^N = (\mathcal{L}^{\frac{1}{M}})^M = \mathcal{L}.$$

Acting on free function, these two fraction powers can be seen as two different locally expansions around zero and infinity respectively. It was stressed that $\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are two different operators even if $N = M$ ($N, M \geq 2$) in [1] due to two different dressing operators. They can also be expressed as following

$$\mathcal{L}^{\frac{1}{N}} = \mathcal{P}_L \Lambda \mathcal{P}_L^{-1}, \quad \mathcal{L}^{\frac{1}{M}} = \mathcal{P}_R \Lambda^{-1} \mathcal{P}_R^{-1}.$$

Let us now define the following operators for the generators of the BTH flows,

$$(3.24) \quad B_{\gamma,n} := \begin{cases} \mathcal{L}^{n+1-\frac{\alpha-1}{N}} & \text{if } \gamma = \alpha = 1, 2, \dots, N \\ \mathcal{L}^{n+1+\frac{\beta}{M}} & \text{if } \gamma = \beta = -M+1, \dots, -1, 0, \end{cases}$$

Definition 3.1. *Bigraded Toda hierarchy (BTH) in the Lax representation is given by the set of infinite number of flows defined by*

$$(3.25) \quad \frac{\partial \mathcal{L}}{\partial t_{\gamma,n}} = \begin{cases} [(B_{\alpha,n})_+, \mathcal{L}], & \text{if } \gamma = \alpha = 1, 2, \dots, N, \\ [-(B_{\beta,n})_-, \mathcal{L}], & \text{if } \gamma = \beta = -M+1, \dots, -1, 0. \end{cases}$$

We need to remark that this kind of definition is equivalent to the definition in [1] which is just a scalar transformation about time variables using gamma function. The original tridiagonal Toda (i.e. Kostant-Toda) hierarchy corresponds to the case with $N = M = 1$.

One can show [2] that the BTH in the Lax representation can be written in the equations of the dressing operators (i.e. the Sato equations):

Theorem 3.2. *The operator \mathcal{L} in (3.20) is a solution to the BTH (3.25) if and only if there is a pair of dressing operators \mathcal{P}_L and \mathcal{P}_R which satisfy the Sato equations,*

$$(3.26) \quad \partial_{\gamma,n} \mathcal{P}_L = -(B_{\gamma,n})_- \mathcal{P}_L, \quad \partial_{\gamma,n} \mathcal{P}_R = (B_{\gamma,n})_+ \mathcal{P}_R$$

for $-M+1 \leq \gamma \leq N$ and $n \geq 0$.

The dressing operators satisfying Sato equations will be called wave operators. By wave operators we will give the definition of tau function for BTH as following.

According to paper [2], a function τ depending only on the dynamical variables t and ϵ is called the **tau-function of BTH** if it provides symbols related to wave operators as following,

$$(3.27) \quad P_L : = 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \dots := \frac{\tau(x, t - [\lambda^{-1}]^N; \epsilon)}{\tau(x, t; \epsilon)},$$

$$(3.28) \quad P_L^{-1} : = 1 + \frac{w'_1}{\lambda} + \frac{w'_2}{\lambda^2} + \dots := \frac{\tau(x + \epsilon, t + [\lambda^{-1}]^N; \epsilon)}{\tau(x + \epsilon, t; \epsilon)},$$

$$(3.29) \quad P_R : = \tilde{w}_0 + \tilde{w}_1 \lambda + \tilde{w}_2 \lambda^2 + \dots := \frac{\tau(x + \epsilon, t + [\lambda]^M; \epsilon)}{\tau(x, t; \epsilon)},$$

$$(3.30) \quad P_R^{-1} : = \tilde{w}'_0 + \tilde{w}'_1 \lambda + \tilde{w}'_2 \lambda^2 + \dots := \frac{\tau(x, t - [\lambda]^M; \epsilon)}{\tau(x + \epsilon, t; \epsilon)},$$

where $[\lambda^{-1}]^N$ and $[\lambda]^M$ are defined by

$$[\lambda^{-1}]_{\gamma,n}^N := \begin{cases} \frac{\lambda^{-N(n+1-\frac{\gamma-1}{N})}}{N(n+1-\frac{\gamma-1}{N})}, & \gamma = N, N-1, \dots, 1, \\ 0, & \gamma = 0, -1, \dots, -(M-1), \end{cases}$$

$$[\lambda]_{\gamma,n}^M := \begin{cases} 0, & \gamma = N, N-1, \dots, 1, \\ \frac{\lambda^{M(n+1+\frac{\beta}{M})}}{M(n+1+\frac{\beta}{M})}, & \gamma = 0, -1, \dots, -(M-1). \end{cases}$$

For a given pair of wave operators the tau-function is unique up to a non-vanishing function factor which is independent of x and $t_{\gamma,n}$ with all $n \geq 0$ and $-M+1 \leq \gamma \leq N$.

Then we get

$$(3.31) \quad P_L := \sum_{n=0}^{\infty} \frac{P_n(-\hat{\partial}_L)\tau(x, t; \epsilon)}{\tau(x, t; \epsilon)} \lambda^{-n}, \quad P_L^{-1} := \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_L)\tau(x + \epsilon, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)} \lambda^{-n},$$

$$(3.32) \quad P_R := \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_R)\tau(x + \epsilon, t; \epsilon)}{\tau(x, t; \epsilon)} \lambda^n, \quad P_R^{-1} := \sum_{n=0}^{\infty} \frac{P_n(-\hat{\partial}_R)\tau(x, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)} \lambda^n,$$

where P_n are the elementary Schur polynomial as defined in (2.10). Here the operators $\hat{\partial}_L$ and $\hat{\partial}_R$ are defined by

$$\hat{\partial}_L = \left\{ \frac{1}{N(n+1-\frac{\alpha-1}{N})} \partial_{t_{\alpha,n}} : 1 \leq \alpha \leq N \right\}$$

$$\hat{\partial}_R = \left\{ \frac{1}{M(n+1+\frac{\beta}{M})} \partial_{t_{\beta,n}} : -M+1 \leq \beta \leq 0 \right\}.$$

The dressing operators \mathcal{P}_L and \mathcal{P}_R can be expressed by function $\tau(x, t; \epsilon)$:

$$(3.33) \quad \mathcal{P}_L = \sum_{n=0}^{\infty} \frac{P_n(-\hat{\partial}_L)\tau(x, t; \epsilon)}{\tau(x, t; \epsilon)} \Lambda^{-n}, \quad \mathcal{P}_L^{-1} = \sum_{n=0}^{\infty} \Lambda^{-n} \frac{P_n(\hat{\partial}_L)\tau(x + \epsilon, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)},$$

$$(3.34) \quad \mathcal{P}_R = \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_R)\tau(x + \epsilon, t; \epsilon)}{\tau(x, t; \epsilon)} \Lambda^n, \quad \mathcal{P}_R^{-1} = \sum_{n=0}^{\infty} \Lambda^n \frac{P_n(-\hat{\partial}_R)\tau(x, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)}.$$

One can then find the explicit form of the coefficients $u_i(x, t)$ of the operator \mathcal{L} in terms of the τ -function using eq.(3.21) as [10, 13],

$$(3.35) \quad u_i(x, t) = \frac{P_{N-i}(\hat{D}_L)\tau(x + (i+1)\epsilon, t; \epsilon) \circ \tau(x, t; \epsilon)}{\tau(x, t; \epsilon) \tau(x + (i+1)\epsilon, t; \epsilon)} = \frac{P_{M+i}(\hat{D}_R)\tau(x + \epsilon, t; \epsilon) \circ \tau(x + i\epsilon, t; \epsilon)}{\tau(x, t; \epsilon) \tau(x + (i+1)\epsilon, t; \epsilon)},$$

where \hat{D}_L and \hat{D}_R are just the Hirota derivatives corresponding to $\hat{\partial}_L$ and $\hat{\partial}_R$ respectively.

The BTH can be also written in the matrix form with the identification of the shift operator Λ as the infinite matrix having zero entries except 1's in the upper diagonal elements and all other functions about x as diagonal infinite matrix [6]. But now we only consider its reduction, i.e. the corresponding semi-infinite matrix form in the following. Then we rewrite the coefficient $u_i(x, t)$ as $u_{i,j}(t)$ and rewrite $\tau(x + \epsilon, t)$ as $\tau_j(t)$. We can find the corresponding semi-infinite matrix forms $\tilde{\mathcal{P}}_L, \tilde{\mathcal{P}}_L^{-1}, \tilde{\mathcal{P}}_R, \tilde{\mathcal{P}}_R^{-1}$ corresponding to $\mathcal{P}_L, \mathcal{P}_L^{-1}, \mathcal{P}_R, \mathcal{P}_R^{-1}$ respectively as following

$$(3.36) \quad \tilde{\mathcal{P}}_L = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{P_1(-\hat{\partial}_L)\tau_1}{\tau_1} & 1 & 0 & 0 & \dots \\ \frac{P_2(-\hat{\partial}_L)\tau_2}{\tau_2} & \frac{P_1(-\hat{\partial}_L)\tau_2}{\tau_2} & 1 & 0 & \dots \\ \frac{P_3(-\hat{\partial}_L)\tau_3}{\tau_3} & \frac{P_2(-\hat{\partial}_L)\tau_3}{\tau_3} & \frac{P_1(-\hat{\partial}_L)\tau_3}{\tau_3} & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$(3.37) \quad \tilde{\mathcal{P}}_L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{P_1(\hat{\partial}_L)\tau_1}{\tau_1} & 1 & 0 & 0 & \dots \\ \frac{P_2(\hat{\partial}_L)\tau_2}{\tau_2} & \frac{P_1(\hat{\partial}_L)\tau_2}{\tau_2} & 1 & 0 & \dots \\ \frac{P_3(\hat{\partial}_L)\tau_3}{\tau_3} & \frac{P_2(\hat{\partial}_L)\tau_3}{\tau_3} & \frac{P_1(\hat{\partial}_L)\tau_3}{\tau_3} & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$(3.38) \quad \tilde{\mathcal{P}}_R = \begin{pmatrix} \frac{\tau_1}{\tau_0} & \frac{P_1(\hat{\partial}_R)\tau_1}{\tau_0} & \frac{P_2(\hat{\partial}_R)\tau_1}{\tau_0} & \frac{P_3(\hat{\partial}_R)\tau_1}{\tau_0} & \dots \\ 0 & \frac{\tau_2}{\tau_1} & \frac{P_1(\hat{\partial}_R)\tau_2}{\tau_1} & \frac{P_2(\hat{\partial}_R)\tau_2}{\tau_1} & \dots \\ 0 & 0 & \frac{\tau_3}{\tau_2} & \frac{P_1(\hat{\partial}_R)\tau_3}{\tau_2} & \dots \\ 0 & 0 & 0 & \frac{\tau_4}{\tau_3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$(3.39) \quad \tilde{\mathcal{P}}_R^{-1} = \begin{pmatrix} \frac{\tau_0}{\tau_1} & \frac{P_1(-\hat{\partial}_R)\tau_1}{\tau_2} & \frac{P_2(-\hat{\partial}_R)\tau_2}{\tau_3} & \frac{P_3(-\hat{\partial}_R)\tau_3}{\tau_4} & \dots \\ 0 & \frac{\tau_1}{\tau_2} & \frac{P_1(-\hat{\partial}_R)\tau_2}{\tau_3} & \frac{P_2(-\hat{\partial}_R)\tau_3}{\tau_4} & \dots \\ 0 & 0 & \frac{\tau_2}{\tau_3} & \frac{P_1(-\hat{\partial}_R)\tau_3}{\tau_4} & \dots \\ 0 & 0 & 0 & \frac{\tau_3}{\tau_4} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

After the following transformation $u_{i,j} = a_{j,j+i}$, the matrix representation of \mathcal{L} can be expressed by $(a_{i,j})_{i,j \geq 1}$ with

$$(3.40) \quad a_{i,j}(t) = \frac{P_{i-j+N}(\hat{D}_L)\tau_j \circ \tau_{i-1}}{\tau_{i-1}\tau_j} = \frac{P_{j-i+M}(\hat{D}_R)\tau_i \circ \tau_{j-1}}{\tau_{i-1}\tau_j}.$$

Note here that those expressions immediately imply

$$a_{i,j} = 0, \quad \text{if } j < -M + i \quad \text{or} \quad j > N + i.$$

That is, the Lax matrix \mathcal{L} has the (N, M) band structure.

As an example, we will give some concrete results on $(2, 2)$ -BTH in the following subsection from which we can see some general patten of (N, M) -BTH.

3.1. Example of the $(2, 2)$ -BTH. Let us summarize this section by taking the $(2, 2)$ -BTH. The Lax operator is

$$(3.41) \quad L = \Lambda^2 + u_1\Lambda + u_0 + u_{-1}\Lambda^{-1} + u_{-2}\Lambda^{-2}.$$

Then there will be two different fraction power of L , denoted as $L_N^{\frac{1}{2}}$ and $L_M^{\frac{1}{2}}$ respectively as following form

$$(3.42) \quad L_N^{\frac{1}{2}} = \Lambda + a_0 + a_{-1}\Lambda^{-1} + a_{-2}\Lambda^{-2} + \dots,$$

$$(3.43) \quad L_M^{\frac{1}{2}} = a'_{-1}\Lambda^{-1} + a'_0 + a'_1\Lambda + a'_2\Lambda^2 + \dots$$

We can get some relations of $\{a_i; i \leq 0\}, \{a'_j; j \geq -1\}$ with $\{u_i; -M \leq i \leq N-1\}$ as following

$$(3.44) \quad a_0(x) = (1 + \Lambda)^{-1}u_1(x), \quad a'_{-1} = e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x)}.$$

Then by Lax equation, we get the $t_{2,0}$ flow of (2,2)-BTH

$$(3.45) \quad \partial_{2,0}L = [\Lambda + (1 + \Lambda)^{-1}u_1(x), L]$$

which correspond to

$$(3.46) \quad \begin{cases} \partial_{2,0}u_1(x) &= u_0(x + \epsilon) - u_0(x) + u_1(x)(1 - \Lambda)(1 + \Lambda)^{-1}u_1(x) \\ \partial_{2,0}u_0(x) &= u_{-1}(x + \epsilon) - u_{-1}(x) \\ \partial_{2,0}u_{-1}(x) &= u_{-2}(x + \epsilon) - u_{-2}(x) + u_{-1}(x)(1 - \Lambda^{-1})(1 + \Lambda)^{-1}u_1(x) \\ \partial_{2,0}u_{-2}(x) &= u_{-2}(x)(1 - \Lambda^{-2})(1 + \Lambda)^{-1}u_1(x). \end{cases}$$

From eqs.(3.46), we can find the equations have infinite terms because of $(1 + \Lambda)^{-1}$ which comes from the fraction power of the Lax operator. Just like the method in [17], for avoiding infinite sums, we use auxiliary function $a_0(x)$ with which we can rewrite eq.(3.46) as

$$(3.47) \quad \begin{cases} \partial_{2,0}u_1(x) &= u_0(x + \epsilon) - u_0(x) + u_1(x)(a_0(x) - a_0(x + \epsilon)) \\ \partial_{2,0}u_0(x) &= u_{-1}(x + \epsilon) - u_{-1}(x) \\ \partial_{2,0}u_{-1}(x) &= u_{-2}(x + \epsilon) - u_{-2}(x) + u_{-1}(x)(a_0(x) - a_0(x - \epsilon)) \\ \partial_{2,0}u_{-2}(x) &= u_{-2}(x)(a_0(x) - a_0(x - 2\epsilon)) \\ \partial_{2,0}a_0(x + \epsilon) + \partial_{2,0}a_0(x) &= u_0(x + \epsilon) - u_0(x) + u_1(x)(a_0(x) - a_0(x + \epsilon)). \end{cases}$$

The $t_{1,0}$ flow will have finite terms as following because it does not use the fraction power of Lax operator L ,

$$(3.48) \quad \partial_{1,0}L = [\Lambda^2 + u_1\Lambda + u_0, L]$$

which correspond to

$$(3.49) \quad \begin{cases} \partial_{1,0}u_1(x) &= u_{-1}(x + 2\epsilon) - u_{-1}(x) \\ \partial_{1,0}u_0(x) &= u_{-2}(x + 2\epsilon) - u_{-2}(x) + u_1(x)u_{-1}(x + \epsilon) - u_{-1}(x)u_1(x - \epsilon) \\ \partial_{1,0}u_{-1}(x) &= u_1(x)u_{-2}(x + \epsilon) - u_{-2}(x)u_1(x - 2\epsilon) + u_{-1}(x)(u_0(x) - u_0(x - \epsilon)) \\ \partial_{1,0}u_{-2}(x) &= u_{-2}(x)(u_0(x) - u_0(x - 2\epsilon)). \end{cases}$$

For $t_{-1,0}$ flow, equations will also be complicated because of another fraction power of L . The equation is

$$(3.50) \quad \partial_{-1,0}L = -[e^{(1+\Lambda^{-1})^{-1} \log u_{-2}} \Lambda^{-1}, L]$$

which corresponds to

$$(3.51) \quad \begin{cases} \partial_{-1,0}u_1(x) &= e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x+2\epsilon)} - e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x)} \\ \partial_{-1,0}u_0(x) &= u_1(x)e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x+\epsilon)} - e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x)}u_1(x - \epsilon) \\ \partial_{-1,0}u_{-1}(x) &= e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x)}(u_0(x) - u_0(x - \epsilon)) \\ \partial_{-1,0}u_{-2}(x) &= u_{-1}(x)e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x-\epsilon)} - e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x)}u_{-1}(x - \epsilon). \end{cases}$$

From eqs.(3.51), we can find the equations have finite terms but every term has infinite multiplication because of factor $e^{(1+\Lambda^{-1})^{-1} \log u_{-2}(x)}$ which comes from the square root of Lax operator. Similarly for avoiding infinite multiplication, we use auxiliary function $a'_{-1}(x)$ with which we can rewrite eq.(3.51) as

$$(3.52) \quad \begin{cases} \partial_{-1,0} u_1(x) &= a'_{-1}(x+2\epsilon) - a'_{-1}(x) \\ \partial_{-1,0} u_0(x) &= u_1(x)a'_{-1}(x+\epsilon) - a'_{-1}(x)u_1(x-\epsilon) \\ \partial_{-1,0} u_{-1}(x) &= a'_{-1}(x)(u_0(x) - u_0(x-\epsilon)) \\ \partial_{-1,0} u_{-2}(x) &= u_{-1}(x)a'_{-1}(x-\epsilon) - a'_{-1}(x)u_{-1}(x-\epsilon) \\ \partial_{-1,0} a'_{-1}(x) &= u_{-1}(x). \end{cases}$$

Infinite sums or infinite multiplications are important properties of BTH because of nonlocal operators. For finite Lax matrix, it is not easy to construct its fraction power. That is why we do not use fraction power of finite matrix to give Lax equations.

4. EQUIVALENCE BETWEEN (N, M) -BTH AND (M, N) -BTH

In this section, we prove that there is an equivalence between (N, M) -BTH and (M, N) -BTH in three ways. One is to prove the equivalence in the Hirota bilinear identities basing on [2]. The second one is for equivalence in specific Hirota bilinear equations. At last, we will prove the equivalence between their Lax equations using transformation in [14]. To see the equivalence clearly, one explicit example, i.e. equivalence between $(1, 2)$ -BTH and $(2, 1)$ -BTH in Lax equations under transformation will be shown in detail.

4.1. Equivalence in the Hirota bilinear identities. Firstly after denoting $\tau(x, t)$ as $\tau(x - \frac{\epsilon}{2}, t)$, we get the following Hirota bilinear identity [2], i.e. for each $m \in \mathbb{Z}$, $r \in \mathbb{N}$,

$$(4.1) \quad \begin{aligned} & \text{Res}_\lambda \left\{ \lambda^{Nr+m-1} \tau(x, t - [\lambda^{-1}]^N) \times \tau(x - (m-1)\epsilon, t' + [\lambda^{-1}]^N) e^{\xi_L(\lambda, t-t')} \right\} \\ &= \text{Res}_\lambda \left\{ \lambda^{-Mr+m-1} \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x - m\epsilon, t' - [\lambda]^M) e^{-\xi_R(\lambda^{-1}, t-t')} \right\}, \end{aligned}$$

where

$$\begin{aligned} \xi_L(\lambda, t) &= \sum_{n \geq 0} \sum_{\alpha=1}^N \lambda^{N(n+1-\frac{\alpha-1}{N})} t_{\alpha, n}, \\ \xi_R(\lambda, t) &= \sum_{n \geq 0} \sum_{\beta=-M+1}^0 \lambda^{M(n+1+\frac{\beta}{M})} t_{\beta, n}. \end{aligned}$$

The most important property that BTH has is that there are Nr and Mr in both sides of HBEs (4.1). These two terms show the principal difference of BTH from the two-dimensional Toda hierarchy, i.e. the constraint (3.21).

Hirota bilinear identity eq.(4.1) can lead to the following identity under the transformation $m \mapsto -m$, $x \mapsto x - m\epsilon$, $\lambda \mapsto \lambda^{-1}$

$$(4.2) \quad \begin{aligned} & \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{-Nr+m} \tau(x - m\epsilon, t - [\lambda]^N) \times \tau(x + \epsilon, t' + [\lambda]^N) e^{\xi'_L(\lambda, t-t')} \right\} \\ &= \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{Mr+m} \tau(x - (m-1)\epsilon, t + [\lambda^{-1}]^M) \times \tau(x, t' - [\lambda^{-1}]^M) e^{-\xi'_R(\lambda^{-1}, t-t')} \right\}, \end{aligned}$$

where

$$\begin{aligned}\xi'_L(\lambda, t - t') &= \sum_{n \geq 0} \sum_{\alpha=1}^N \lambda^{-N(n+1-\frac{\alpha-1}{N})} (t_{\alpha,n} - t'_{\alpha,n}), \\ \xi'_R(\lambda^{-1}, t - t') &= \sum_{n \geq 0} \sum_{\beta=-M+1}^0 \lambda^{M(n+1+\frac{\beta}{M})} (t_{\beta,n} - t'_{\beta,n}).\end{aligned}$$

The eq.(4.2) can be rewritten as following identity after the interchanging of t and t'

$$\begin{aligned}(4.3) \quad & \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{-Nr+m} \tau(x + \epsilon, t + [\lambda]^N) \times \tau(x - m\epsilon, t' - [\lambda]^N) e^{\xi'_L(\lambda, t'-t)} \right\} \\ &= \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{Mr+m} \tau(x, t - [\lambda^{-1}]^M) \times \tau(x - (m-1)\epsilon, t' + [\lambda^{-1}]^M) e^{-\xi'_R(\lambda^{-1}, t'-t)} \right\}\end{aligned}$$

which can be further rewritten as following

$$\begin{aligned}(4.4) \quad & \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{Mr+m} \tau(x, t - [\lambda^{-1}]^M) \times \tau(x - (m-1)\epsilon, t' + [\lambda^{-1}]^M) e^{\xi_R(\lambda, t-t')} \right\} \\ &= \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{-Nr+m} \tau(x + \epsilon, t + [\lambda]^N) \times \tau(x - m\epsilon, t' - [\lambda]^N) e^{-\xi_L(\lambda^{-1}, t-t')} \right\}.\end{aligned}$$

Eq.(4.4) is obviously (M, N) -BTH comparing to eq.(4.1) if we change time variable $t_{\gamma,n}$ to $t_{1-\gamma,n}$, i.e. subscript $L \leftrightarrow R$. Therefore for (M, N) -BTH, Eq.(4.4) is in fact changed into the following equation under transformation $t_{\gamma,n} \rightarrow t_{1-\gamma,n}$

$$\begin{aligned}(4.5) \quad & \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{Mr+m} \tau(x, t - [\lambda^{-1}]^M) \times \tau(x - (m-1)\epsilon, t' + [\lambda^{-1}]^M) e^{\xi_L(\lambda, t-t')} \right\} \\ &= \text{Res}_\lambda \left\{ \lambda^{-1} \lambda^{-Nr+m} \tau(x + \epsilon, t + [\lambda]^N) \times \tau(x - m\epsilon, t' - [\lambda]^N) e^{-\xi_R(\lambda^{-1}, t-t')} \right\}.\end{aligned}$$

Because transforms $m \mapsto -m, x \mapsto x - m\epsilon, \lambda \mapsto \lambda^{-1}$ do not change the equation itself. Therefore we can say that the (N, M) -BTH is equivalent to the (M, N) -BTH under the transformation $t_{\gamma,n} \rightarrow t_{1-\gamma,n}$.

4.2. Equivalence in the Hirota bilinear equations. The bilinear identity eq.(4.1) of the BTH can be equivalently expressed as [2]

$$\begin{aligned}(4.6) \quad & \text{Res}_\lambda \left\{ \lambda^{Nr+m-1} \tau_{j-(m-1)}(t + y + [\lambda^{-1}]^N) \tau_j(t - y - [\lambda^{-1}]^N) e^{\xi_L(\lambda, -2y)} \right\} \\ &= \text{Res}_\lambda \left\{ \lambda^{-Mr+m-1} \tau_{j+1}(t - y + [\lambda]^M) \tau_{j-m}(t + y - [\lambda]^M) e^{\xi_R(\lambda^{-1}, 2y)} \right\}.\end{aligned}$$

To be specific, we will give the equivalence from the concrete Hirota equations between (N, M) -BTH and (M, N) -BTH as following which will also be used to derive specific primary Hirota equations of BTH in the next section.

For (N, M) -BTH, in term with $\prod y_{\alpha_1, l_1}^{k_1} y_{\alpha_2, l_2}^{k_2} \cdots y_{\alpha_s, l_s}^{k_s} \prod y_{\beta_1, l'_1}^{k'_1} y_{\beta_2, l'_2}^{k'_2} \cdots y_{\beta_p, l'_p}^{k'_p}, -M+1 \leq \beta_i \leq 0, 1 \leq \alpha_i \leq N$, the Hirota equation is as

$$\begin{aligned}& \prod_{v=1}^s \frac{(-D_{\alpha_v, l_v})^{k_v}}{k_v!} \left(\sum_{i'_1=0}^{k'_1} \cdots \sum_{i'_p=0}^{k'_p} \prod_{q=1}^p \frac{(-D_{\beta_q, l'_q})^{i'_q}}{i'_q!} \frac{2^{k'_q-i'_q}}{(k'_q-i'_q)!} P_{\sum_{q=1}^p M(l'_q+1+\frac{\beta_q}{M})(k'_q-i'_q)+Mr-m}(\hat{D}_R) \right) \tau_{n+1} \tau_{n-m} \\ &= \prod_{v=1}^p \frac{D_{\beta_v, l'_v}^{k'_v}}{k'_v!} \left(\sum_{i_1=0}^{k_1} \cdots \sum_{i_s=0}^{k_s} \prod_{q=1}^s \frac{(D_{\alpha_q, l_q})^{i_q}}{i_q!} \frac{(-2)^{k_q-i_q}}{(k_q-i_q)!} P_{\sum_{q=1}^s N(l_q+1-\frac{\alpha_q-1}{N})(k_q-i_q)+Nr+m}(\hat{D}_L) \right) \tau_{n-m+1} \tau_n,\end{aligned}$$

(4.7)

where P_k are Schur polynomial as defined in (2.10). After the transformation $m \mapsto -m, n \mapsto n - m$, the identity eq.(4.7) becomes

$$\begin{aligned} & \prod_{v=1}^p \frac{D_{\beta_v, l'_v}^{k'_v}}{k'_v!} \left(\sum_{i_1=0}^{k_1} \cdots \sum_{i_s=0}^{k_s} \prod_{q=1}^s \frac{(D_{\alpha_q, l_q})^{i_q}}{i_q!} \frac{(-2)^{k_q-i_q}}{(k_q-i_q)!} P_{\sum_{q=1}^s N(l_q+1-\frac{\alpha_q-1}{N})(k_q-i_q)+Nr-m}(\hat{D}_L) \right) \tau_{n+1} \tau_{n-m} \\ &= \prod_{v=1}^s \frac{(-D_{\alpha_v, l_v})^{k_v}}{k_v!} \left(\sum_{i'_1=0}^{k'_1} \cdots \sum_{i'_p=0}^{k'_p} \prod_{q=1}^p \frac{(-D_{\beta_q, l'_q})^{i'_q}}{i'_q!} \frac{2^{k'_q-i'_q}}{(k'_q-i'_q)!} P_{\sum_{q=1}^p M(l'_q+1+\frac{\beta_q}{M})(k'_q-i'_q)+Mr+m}(\hat{D}_R) \right) \\ (4.8) \quad & \tau_{n-m+1} \tau_n. \end{aligned}$$

After doing the transformation $D_{\gamma, l} = D'_{1-\gamma, l}$, $-M+1 \leq \gamma \leq N$, eq.(4.8) becomes

$$\begin{aligned} & \prod_{v=1}^p \frac{(D'_{1-\beta_v, l'_v})^{k'_v}}{k'_v!} \left(\sum_{i_1=0}^{k_1} \cdots \sum_{i_s=0}^{k_s} \prod_{q=1}^s \frac{(D'_{1-\alpha_q, l_q})^{i_q}}{i_q!} \frac{(-2)^{k_q-i_q}}{(k_q-i_q)!} P_{\sum_{q=1}^s N(l_q+1+\frac{1-\alpha_q}{N})(k_q-i_q)+Nr-m}(\hat{D}'_R) \right) \\ \tau_{n+1} \tau_{n-m} &= \prod_{v=1}^s \frac{(-D'_{1-\alpha_v, l_v})^{k_v}}{k_v!} \\ (4.9) \quad & \left(\sum_{i_1=0}^{k'_1} \cdots \sum_{i_p=0}^{k'_p} \prod_{q=1}^p \frac{(-D'_{1-\beta_q, l'_q})^{i'_q}}{i'_q!} \frac{2^{k'_q-i'_q}}{(k'_q-i'_q)!} P_{\sum_{q=1}^p M(l'_q+1+\frac{\beta}{M})(k'_q-i'_q)+Mr+m}(\hat{D}'_L) \right) \tau_{n-m+1} \tau_n. \end{aligned}$$

which is the term with $\prod y_{1-\beta_1, l'_1}^{k'_1} y_{1-\beta_2, l'_2}^{k'_2} \cdots y_{1-\beta_p, l'_p}^{k'_p} \prod y_{1-\alpha_1, l_1}^{k_1} y_{1-\alpha_2, l_2}^{k_2} \cdots y_{1-\alpha_s, l_s}^{k_s}$, $1 \leq 1-\beta_i \leq M$, $-N+1 \leq 1-\alpha_i \leq 0$ for (M, N) -BTH. So there is a correspondence of (N, M) -BTH and (M, N) -BTH in Hirota equations under the meaning of following derivatives' correspondence $D'_{\gamma, l} \leftrightarrow D_{1-\gamma, l}$, $-M+1 \leq \gamma \leq N$, i.e. $D'_L \leftrightarrow D_R$, $D'_R \leftrightarrow D_L$.

Because (N, M) -BTH and (M, N) -BTH are equivalent, we can only consider BTH in the case of $N \leq M$ later.

4.3. Equivalence in the Lax equations. Using the gauge transformation and linear transformation mentioned in [14], we can prove the equivalence between (N, M) -BTH and (M, N) -BTH under the meaning of Lax equations.

Firstly we need to introduce the following proposition basing on Theorem 6 in [14].

Proposition 4.1. *If \mathcal{L} satisfies Lax equations (3.25) and let $\Phi = \Phi(u, t)$ satisfies condition*

$$(4.10) \quad \partial_{t_{\gamma, n}} \Phi = (B_{\gamma, n})_0 \Phi, \quad -M+1 \leq \gamma \leq N, n \geq 0,$$

(where subscript 0 denotes the projection to term of Λ^0), then $\tilde{\mathcal{L}} = \Phi^{-1} \mathcal{L} \Phi$ satisfies the hierarchy

$$(4.11) \quad \partial_{t_{\gamma, n}} \tilde{\mathcal{L}} = [(\tilde{B}_{\gamma, n})_{\geq 1}, \tilde{\mathcal{L}}], \quad -M+1 \leq \gamma \leq N, n \geq 0,$$

where

$$(4.12) \quad \tilde{B}_{\gamma, n} = \Phi^{-1} B_{\gamma, n} \Phi.$$

Proof. Because of (3.25) and (4.10), the following calculation holds

$$\partial_{t_{\gamma, n}} \tilde{\mathcal{L}} - [(\tilde{B}_{\gamma, n})_{\geq 1}, \tilde{\mathcal{L}}] = \partial_{t_{\gamma, n}} (\Phi^{-1} \mathcal{L} \Phi) - \Phi^{-1} [(B_{\gamma, n})_{\geq 1}, \mathcal{L}] \Phi$$

$$\begin{aligned}
&= \Phi^{-1}(\partial_{t_{\gamma,n}} \mathcal{L} - [(B_{\gamma,n})_+, \mathcal{L}])\Phi - [\Phi^{-1}(\partial_{t_{\gamma,n}} \Phi - (B_{\gamma,n})_0 \Phi), \tilde{\mathcal{L}}] \\
&= 0.
\end{aligned}$$

Then we finished the proof of this proposition. \square

Proposition 4.1 tells us that the gauge transformation from Theorem 6 in [14] can be extended on the fractional powers of Lax operators. Now let us introduce the following notation for anti-involution map “ \dagger ”: If \mathcal{L} is as form (3.20), then

$$\begin{aligned}
\mathcal{L}^\dagger : &= \Lambda^M u_{-M}(x) + \Lambda^{M-1} u_{-M+1}(x) + \cdots + \Lambda^{-N+1} u_{N-1}(x) + \Lambda^{-N} \\
&= u_{-M}(x + M\epsilon) \Lambda^M + u_{-M+1}(x + (M+1)\epsilon) \Lambda^{M-1} + \cdots + u_{N-1}(x - (N-1)\epsilon) \Lambda^{-N+1} + \Lambda^{-N}.
\end{aligned}$$

By calculation, one can prove the following two lemmas similar as [14].

Lemma 4.2. *For $-M+1 \leq \gamma \leq N, n \geq 0$, following identity holds for any integer k*

$$(4.13) \quad ((B_{\gamma,n})_{\geq k})^\dagger = ((B_{\gamma,n})^\dagger)_{\leq -k}.$$

Using above lemma, we can prove the following lemma directly.

Lemma 4.3. *$\partial_{t_{\gamma,n}} \mathcal{L} = [(B_{\gamma,n})_{\geq 0}, \mathcal{L}]$ can lead to $\partial_{t_{\gamma,n}} \mathcal{L}^\dagger = [((B_{\gamma,n})^\dagger)_{\geq 1}, \mathcal{L}^\dagger]$ and $\partial_{t_{\gamma,n}} \mathcal{L} = [(B_{\gamma,n})_{\geq 1}, \mathcal{L}]$ can lead to $\partial_{t_{\gamma,n}} \mathcal{L}^\dagger = [((B_{\gamma,n})^\dagger)_{\geq 0}, \mathcal{L}^\dagger]$.*

By above two lemmas and Proposition 4.1, we can prove the following important theorem.

Theorem 4.4. *Lax equation (3.25) of (N, M) -BTH with Lax operator*

$$(4.14) \quad \mathcal{L}_{N,M} = \Lambda^N + u_{N-1} \Lambda^{N-1} + \cdots + u_0 + \cdots + u_{-M} \Lambda^{-M}$$

is equivalent to (M, N) -BTH with Lax operator

$$(4.15) \quad \mathcal{L}_{M,N} = \Lambda^M + \tilde{u}_{M-1} \Lambda^{M-1} + \cdots + \tilde{u}_0 + \cdots + \tilde{u}_{-N} \Lambda^{-N}$$

under Miura map: $\tilde{u}_j(x, t) = u_{-j}(x + j\epsilon) e^{\frac{1-\Lambda^j}{1-\Lambda^{-M}} u_{-M}(x, t)}$.

Proof. For Lax operator $\mathcal{L}_{N,M} = \Lambda^N + u_{N-1} \Lambda^{N-1} + \cdots + u_0 + \cdots + u_{-M} \Lambda^{-M}$ of (N, M) -BTH which satisfies Lax equations (3.25), one can choose $\Psi = \Psi(u, t) = e^{(1-\Lambda^{-M})^{-1} u_{-M}(x, t)}$ which satisfies condition

$$(4.16) \quad \partial_{t_{\gamma,n}} \Psi = (B_{\gamma,n})_0 \Psi, \quad -M+1 \leq \gamma \leq N, n \geq 0.$$

Then $\tilde{\mathcal{L}}_{N,M} = \Psi^{-1} \mathcal{L}_{N,M} \Psi = \bar{u}_N \Lambda^N + \bar{u}_{N-1} \Lambda^{N-1} + \cdots + \bar{u}_0 + \cdots + \Lambda^{-M}$ satisfies the hierarchy

$$(4.17) \quad \partial_{t_{\gamma,n}} \tilde{\mathcal{L}}_{N,M} = [(\tilde{B}_{\gamma,n})_{\geq 1}, \tilde{\mathcal{L}}_{N,M}], \quad -M+1 \leq \gamma \leq N, n \geq 0,$$

where

$$(4.18) \quad \tilde{B}_{\gamma,n} = \Psi^{-1} B_{\gamma,n} \Psi, \quad \bar{u}_i(x) = \Psi^{-1}(x) u_i(x) \Psi(x + i\epsilon).$$

Using Lemma 4.3, one can derive

$$(4.19) \quad \partial_{t_{\gamma,n}} \tilde{\mathcal{L}}_{N,M}^\dagger = [((\tilde{B}_{\gamma,n})^\dagger)_{\geq 0}, \tilde{\mathcal{L}}_{N,M}^\dagger], \quad -M+1 \leq \gamma \leq N, n \geq 0,$$

One can choose Lax operator $\mathcal{L}_{M,N} = \tilde{\mathcal{L}}_{N,M}^\dagger = \Lambda^M + \tilde{u}_{M-1} \Lambda^{M-1} + \cdots + \tilde{u}_0 + \cdots + \tilde{u}_{-N} \Lambda^{-N}$ of (M, N) -BTH with

$$\tilde{u}_j(x) = \bar{u}_{-j}(x + j\epsilon) = \Psi^{-1}(x + j\epsilon) u_{-j}(x + j\epsilon) \Psi(x) = u_{-j}(x + j\epsilon) e^{\frac{1-\Lambda^j}{1-\Lambda^{-M}} u_{-M}(x, t)}.$$

The (M, N) -BTH with this Lax matrix will be equivalent to the original (N, M) -BTH. \square

To be illustrative, we will give some simplest concrete examples of equivalence in the next subsection which includes nonlocal flows of (2, 1)-BTH and (1, 2)-BTH. Here we will also consider the interpolated form of BTH. At this time, we will use a powerful tool called gauge transformation to prove that equivalence.

4.4. Equivalence between (1, 2)-BTH and (2, 1)-BTH. Using Proposition 4.1, we will see the equivalence between (1, 2)-BTH and (2, 1)-BTH in detail. Here we only give the primary flows of them to see the equivalence.

(1, 2)-BTH: The Lax operator of (1, 2)-BTH is as following

$$(4.20) \quad \mathcal{L}_{1,2} = \Lambda + u_0 + u_{-1}\Lambda^{-1} + u_{-2}\Lambda^{-2}.$$

(1, 2)-BTH has the following primary equations

$$(4.21) \quad \partial_{1,0}\mathcal{L}_{1,2} = [\Lambda + u_0, \mathcal{L}_{1,2}],$$

and

$$(4.22) \quad \partial_{-1,0}\mathcal{L}_{1,2} = -[e^{(1+\Lambda^{-1})^{-1}\log u_{-2}}\Lambda^{-1}, \mathcal{L}_{1,2}],$$

which further lead to

$$(4.23) \quad \begin{cases} \partial_{-1,0}u_0(x) = e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x+\epsilon)} - e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x)}, \\ \partial_{-1,0}u_{-1}(x) = e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x)}(u_0(x) - u_0(x - \epsilon)), \\ \partial_{-1,0}u_{-2}(x) = u_{-1}(x)e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x-\epsilon)} - e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x)}u_{-1}(x - \epsilon). \end{cases}$$

and

$$(4.24) \quad \begin{cases} \partial_{1,0}u_0(x) = u_{-1}(x + \epsilon) - u_{-1}(x), \\ \partial_{1,0}u_{-1}(x) = u_{-2}(x + \epsilon) - u_{-2}(x) + u_{-1}(x)(u_0(x) - u_0(x - \epsilon)), \\ \partial_{1,0}u_{-2}(x) = u_{-2}(x)(u_0(x) - u_0(x - 2\epsilon)). \end{cases}$$

(2, 1)-BTH: The Lax operator of (2, 1)-BTH is as following

$$(4.25) \quad \mathcal{L}_{2,1} = \Lambda^2 + \bar{u}_1\Lambda + \bar{u}_0 + \bar{u}_{-1}\Lambda^{-1}.$$

The equations (3.25) in this case are as follows

$$(4.26) \quad \partial_{2,0}\mathcal{L}_{2,1} = [\Lambda + (1 + \Lambda)^{-1}\bar{u}_1(x), \mathcal{L}_{2,1}],$$

$$(4.27) \quad \partial_{1,0}\mathcal{L}_{2,1} = [\Lambda^2 + \bar{u}_1\Lambda + \bar{u}_0, \mathcal{L}_{2,1}],$$

which further lead to the following concrete equations

$$(4.28) \quad \begin{cases} \partial_{2,0}\bar{u}_1(x) = \bar{u}_1(x + \epsilon) - \bar{u}_1(x) + \bar{u}_1(x)(1 - \Lambda)(1 + \Lambda)^{-1}\bar{u}_1(x), \\ \partial_{2,0}\bar{u}_0(x) = \bar{u}_{-1}(x + \epsilon) - \bar{u}_{-1}(x), \\ \partial_{2,0}\bar{u}_{-1}(x) = \bar{u}_{-1}(x)(1 - \Lambda^{-1})(1 + \Lambda)^{-1}\bar{u}_1(x), \end{cases}$$

$$(4.29) \quad \begin{cases} \partial_{1,0}\bar{u}_1(x) = \bar{u}_{-1}(x + 2\epsilon) - \bar{u}_{-1}(x), \\ \partial_{1,0}\bar{u}_0(x) = \bar{u}_{-2}(x + 2\epsilon) - \bar{u}_{-2}(x) + \bar{u}_1(x)\bar{u}_{-1}(x + \epsilon) - \bar{u}_{-1}(x)\bar{u}_1(x - \epsilon), \\ \partial_{1,0}\bar{u}_{-1}(x) = \bar{u}_{-1}(x)(\bar{u}_0(x) - \bar{u}_0(x - \epsilon)). \end{cases}$$

It seems that eq.(4.28) and eq.(4.29) are quite different from eq.(4.23) and eq.(4.24) respectively. In fact after doing gauge transformation on (2, 1)-BTH as following, we can find the equivalent relation between these equations.

Now we consider function ϕ has form

$$(4.30) \quad \phi = e^{(1-\Lambda)^{-1} \log \bar{u}_{-1}(x)},$$

then $\hat{\mathcal{L}}_{2,1} := \phi^{-1} \mathcal{L}_{2,1} \phi$ will have form

$$(4.31) \quad \hat{\mathcal{L}}_{2,1} = v_2 \Lambda^2 + v_1 \Lambda + v_0 + \Lambda^{-1}.$$

The relation of $v_i (0 \leq i \leq 2)$ and $\bar{u}_i (-1 \leq i \leq 1)$ are like

$$(4.32) \quad v_2 = \phi^{-1} \phi(x+2\epsilon), \quad v_1 = \phi^{-1} u_1 \phi(x+\epsilon), \quad v_0 = u_0.$$

Therefore we get following new flows on new Lax operator $\hat{\mathcal{L}}_{2,1}$ using Proposition 4.1

$$(4.33) \quad \partial_{2,0} \hat{\mathcal{L}}_{2,1} = [e^{(1+\Lambda)^{-1} \log v_2(x)} \Lambda, \hat{\mathcal{L}}_{2,1}],$$

$$(4.34) \quad \partial_{1,0} \hat{\mathcal{L}}_{2,1} = [v_2 \Lambda^2 + v_1 \Lambda^1, \hat{\mathcal{L}}_{2,1}],$$

which further leads to

$$(4.35) \quad \begin{cases} \partial_{2,0} v_0(x) = e^{(1+\Lambda)^{-1} \log v_2(x)} - e^{(1+\Lambda)^{-1} \log v_2(x-\epsilon)}, \\ \partial_{2,0} v_1(x) = e^{(1+\Lambda)^{-1} \log v_2(x)} (v_0(x+\epsilon) - v_0(x)), \\ \partial_{2,0} v_2(x) = v_1(x+\epsilon) e^{(1+\Lambda)^{-1} \log v_2(x)} - e^{(1+\Lambda)^{-1} \log v_2(x+\epsilon)} v_1(x). \end{cases}$$

and

$$(4.36) \quad \begin{cases} \partial_{1,0} v_0(x) = v_1(x) - v_1(x-\epsilon), \\ \partial_{1,0} v_1(x) = v_2(x) - v_2(x-\epsilon) + v_1(x) (v_0(x+\epsilon) - v_0(x)), \\ \partial_{1,0} v_2(x) = v_2(x) (v_0(x+2\epsilon) - v_0(x)). \end{cases}$$

Comparing eq.(4.35), eq.(4.36) with eq.(4.23) and eq.(4.24), we can find these two pairs of flows are equivalent under Miura map $u_j = v_{-j}(x+j\epsilon) (0 \leq j \leq 2)$. This proves the equivalence between (1,2)-BTH and (2,1)-BTH.

Using bilinear identities got by comparing every term of Hirota bilinear identities of BTH in this section, in the next section we will derive all primary Hirota equations of BTH to see its inner structure.

5. HIROTA EQUATIONS AND SOLUTIONS OF THE BTH

Hirota bilinear equations (HBEs) are central object in Sato theory. From HBEs, we can derive the structure of solution. This is a great motivation for us to consider HBEs of the BTH. The Hirota bilinear equations of the BTH can be derived from (4.6) which comes from HBEs in [2]. In particular, the following proposition will list all the Hirota equations for the primary variables, i.e. $t_{\gamma,n}$ with $n = 0$.

Proposition 5.1. *For (N, M) -BTH, we have the following identities for primary derivatives which are equivalent to all the primary Hirota equations.*

$$(5.1) \quad \left(D_{\beta,0} - P_{M+\beta}(\hat{D}_R) \right) \tau_{n+1} \circ \tau_n = 0,$$

$$(5.2) \quad \left(D_{\beta,0} D_{-M+1,0} - 2P_{M+\beta+1}(\hat{D}_R) \right) \tau_n \circ \tau_n = 0,$$

$$(5.3) \quad D_{\beta,0} D_{N,0} \tau_n \circ \tau_n = 2P_{M+\beta-1}(\hat{D}_R) \tau_{n+1} \circ \tau_{n-1},$$

$$(5.4) \quad D_{\alpha,0} D_{-M+1,0} \tau_n \circ \tau_n = 2P_{N-\alpha}(\hat{D}_L) \tau_{n+1} \circ \tau_{n-1}$$

$$(5.5) \quad \left(D_{\alpha,0} D_{N,0} - 2P_{N-\alpha+2}(\hat{D}_L) \right) \tau_n \circ \tau_n = 0,$$

$$(5.6) \quad \left(D_{\alpha,0} - P_{N-\alpha+1}(\hat{D}_L) \right) \tau_{n+1} \circ \tau_n = 0.$$

where P_k are Schur polynomial as defined in (2.10).

Proof. In eq.(4.6), for term $y_{\beta,l}^k$, $-M+1 \leq \beta \leq 0$ in (N, M) -BTH, following Hirota equation holds

$$(5.7) \quad \left(\sum_{i=0}^k \frac{(-D_{\beta,l})^i}{i!} \frac{2^{k-i}}{(k-i)!} P_{M(l+1+\frac{\beta}{M})(k-i)+Mr-m}(\hat{D}_R) \right) \tau_{n+1} \tau_{n-m} = \frac{(D_{\beta,l})^k}{k!} P_{Nr+m}(\hat{D}_L) \tau_{n-m+1} \tau_n,$$

which can also be got from eq.(4.7).

For $y_{\alpha,l}^k$, $1 \leq \alpha \leq N$ in (N, M) -BTH, following Hirota equation holds

$$(5.8) \quad \left(\sum_{i=0}^k \frac{(-D_{\alpha,l})^i}{i!} \frac{2^{k-i}}{(k-i)!} P_{N(l+1-\frac{\alpha-1}{N})(k-i)+Nr+m}(\hat{D}_L) \right) \tau_{n-m+1} \tau_n = \frac{(D_{\alpha,l})^k}{k!} P_{Mr-m}(\hat{D}_R) \tau_{n+1} \tau_{n-m}.$$

For $y_{\beta,0}$, $-M+1 \leq \beta \leq 0$ in (N, M) -BTH, the following Hirota equation holds

$$\left(\sum_{i=0}^1 \frac{(-D_{\beta,l})^i}{i!} \frac{2^{1-i}}{(1-i)!} P_{M(l+1+\frac{\beta}{M})(1-i)+Mr-m}(\hat{D}_R) \right) \tau_{n+1} \tau_{n-m} = D_{\beta,0} P_{Nr+m}(\hat{D}_L) \tau_{n-m+1} \tau_n,$$

which is

$$(2P_{M(1+\frac{\beta}{M})+Mr-m}(\hat{D}_R) - D_{\beta,0} P_{Mr-m}(\hat{D}_R)) \tau_{n+1} \tau_{n-m} = D_{\beta,0} P_{Nr+m}(\hat{D}_L) \tau_{n-m+1} \tau_n.$$

Set $r=0, m=-1$, for term with $y_{\beta,0}$, $-M+1 \leq \beta \leq 0$, we get

$$(2P_{M+\beta+1}(\hat{D}_R) - D_{\beta,0} D_{-M+1,0}) \tau_{n+1} \tau_{n+1} = 0.$$

When $r=0, m=0$,

for term with $y_{\beta,0}$, $-M+1 \leq \beta \leq 0$ in (N, M) -BTH, we get

$$(5.9) \quad (P_{M+\beta}(\hat{D}_R) - D_{\beta,0}) \tau_{n+1} \tau_n = 0.$$

Set $r=0, m=1$, for term $y_{\beta,0}$, $-M+1 \leq \beta \leq 0$ in (N, M) -BTH, we get

$$2P_{M+\beta-1}(\hat{D}_R) \tau_{n+1} \tau_{n-1} = D_{\beta,0} D_{N,0} \tau_n \tau_n.$$

For term $y_{\alpha,0}$, $1 \leq \alpha \leq N$ in (N, M) -BTH, following Hirota equation holds

$$(5.10) \quad \left(\sum_{i=0}^1 \frac{(-D_{\alpha,l})^i}{i!} \frac{2^{1-i}}{(1-i)!} P_{N(1-\frac{\alpha-1}{N})(1-i)+Nr+m}(\hat{D}_L) \right) \tau_{n-m+1} \tau_n = D_{\alpha,0} P_{Mr-m}(\hat{D}_R) \tau_{n+1} \tau_{n-m},$$

which is

$$(2P_{N(1-\frac{\alpha-1}{N})+Nr+m}(\hat{D}_L) - D_{\alpha,0}P_{Nr+m}(\hat{D}_L))\tau_{n-m+1}\tau_n = D_{\alpha,0}P_{Mr-m}(\hat{D}_R)\tau_{n+1}\tau_{n-m}.$$

For $r = 0, m = -1$, it further leads to

$$2P_{N-\alpha}(\hat{D}_L)\tau_{n+2}\tau_n = D_{\alpha,0}D_{-M+1,0}\tau_{n+1}\tau_{n+1}.$$

Set $r = 0, m = 0$, for term with $y_{\alpha,0}, 1 \leq \alpha \leq N$ in (N, M) -BTH, following equation succeeds

$$(P_{N-(\alpha-1)}(\hat{D}_L) - D_{\alpha,0})\tau_{n+1}\tau_n = 0$$

Set $r = 0, m = 1$, we get equation

$$(\frac{1}{2}D_{\alpha,0}D_{N,0} - P_{N-\alpha+2}(\hat{D}_L))\tau_n\tau_n = 0.$$

You can get the primary Hirota equations from the value of $k = 0, 1$. The other values of k will give higher order derivatives of the hierarchy.

When $k = 0$, the equations eq.(5.7) and eq.(5.8) will give

$$(5.11) \quad P_{Nr+m}(\hat{D}_L)\tau_{n-m+1}\tau_n = P_{Mr-m}(\hat{D}_R)\tau_{n+1}\tau_{n-m}.$$

when $r = 1, m = 0$, eq.(5.11) becomes

$$(5.12) \quad P_N(\hat{D}_L)\tau_{n+1}\tau_n = P_M(\hat{D}_R)\tau_{n+1}\tau_n.$$

when $r = 1, m = -1$, eq.(5.11) becomes

$$(5.13) \quad P_{N-1}(\hat{D}_L)\tau_{n+2}\tau_n = P_{M+1}(\hat{D}_R)\tau_{n+1}\tau_{n+1}.$$

when $r = 1, m = 1$, we get

$$(5.14) \quad P_{N+1}(\hat{D}_L)\tau_n\tau_n = P_{M-1}(\hat{D}_R)\tau_{n+1}\tau_{n-1}.$$

The three equations eq.(5.12), eq.(5.13), eq.(5.14) can be derived from eq.(5.1)-eq.(5.4). These equations discussed above are all the primary Hirota equations. The other values of r and m will include higher derivatives of the hierarchy. \square

From these primary Hirota equations, we can see BTH has ample structure information. We can say BTH contains discrete KP equation (from eq.(5.2) and eq.(5.5)), NLS equations (from eq.(5.1) and eq.(5.6)) and 2-dimensional Toda lattice equation (from eq.(5.3) and eq.(5.4)). Comparing with Hirota equations of the two-dimensional Toda hierarchy we find BTH have more constraints on equations which comes from the equivalence of $\partial_{t_{1,n}}$ and $\partial_{t_{0,n}}$. These information help us to get the solution of BTH which will be given in the next subsection. From all the primary Hirota equations mentioned above, we can get that the solution of BTH should have double-wronskian structure [17]. In paper [17], if we impose the vanishing of ∂_y derivatives on tau functions the molecule equation in fact becomes our $(2, 1)$ -BTH. Using the same method in [17], we proved the double-wronskian solution structure satisfy all the primary Hirota equations of (N, M) -BTH. But later we find there is another much simpler way to get the structure naturally which will be mentioned in the next subsection. So we prefer this simpler way to the way in [17].

5.1. Solutions of the BTH in the semi-infinite matrix. Now we construct tau function for the BTH in the semi-infinite matrix representation, that is, $(a_{i,j})_{i,j \geq 1}$. In order to do this, we first introduce the wave operators \mathcal{W}_L and \mathcal{W}_R associated with the dressing operators \mathcal{P}_L and \mathcal{P}_R ,

$$(5.15) \quad \mathcal{W}_L(x, t, \Lambda) = \mathcal{P}_L(x, t, \Lambda) \circ \exp \left(\sum_{n \geq 0} \sum_{\alpha=1}^N \Lambda^{N(n+1-\frac{\alpha-1}{N})} t_{\alpha,n} \right),$$

$$(5.16) \quad \mathcal{W}_R(x, t, \Lambda) = \mathcal{P}_R(x, t, \Lambda) \circ \exp \left(- \sum_{n \geq 0} \sum_{\beta=-M+1}^0 \Lambda^{-M(n+1+\frac{\beta}{M})} t_{\beta,n} \right).$$

By sato equations (3.26), we can have following identities [2]

$$\begin{aligned} \partial_{\alpha,n} \mathcal{W}_L &:= \begin{cases} (B_{\alpha,n})_+ \mathcal{W}_L, & \alpha = N \dots 1, \\ -(B_{\alpha,n})_- \mathcal{W}_L, & \alpha = 0 \dots -M+1, \end{cases} \\ \partial_{\alpha,n} \mathcal{W}_R &:= \begin{cases} (B_{\alpha,n})_+ \mathcal{W}_R, & \alpha = N \dots 1, \\ -(B_{\alpha,n})_- \mathcal{W}_R, & \alpha = 0 \dots -M+1. \end{cases} \end{aligned}$$

One can then prove that the product $\mathcal{W}_L^{-1} \mathcal{W}_R$ is invariant under all the flows, i.e.

$$\partial_{\gamma,n} (\mathcal{W}_L^{-1} \mathcal{W}_R) = 0.$$

Therefore

$$\mathcal{W}_L^{-1} \mathcal{W}_R(x, t, \Lambda) = \mathcal{W}_L^{-1} \mathcal{W}_R(x, 0, \Lambda) = \mathcal{P}_L^{-1} \mathcal{P}_R(x, 0, \Lambda).$$

This implies

$$(5.17) \quad (\mathcal{P}_L^{-1} \mathcal{P}_R)(t) = \exp \left(\sum_{n \geq 0} \sum_{\alpha=1}^N \Lambda^{N(n+1-\frac{\alpha-1}{N})} t_{\alpha,n} \right) \circ (\mathcal{P}_L^{-1} \mathcal{P}_R)(0) \circ \exp \left(\sum_{n \geq 0} \sum_{\beta=-M+1}^0 \Lambda^{-M(n+1+\frac{\beta}{M})} t_{\beta,n} \right).$$

If we let $\tau_0 = 1, \tau_i = 0 (i \in \mathbb{Z}_-)$; then bi-infinite matrix everywhere will become semi-infinite matrix. Representing the product $\mathcal{P}_L^{-1} \mathcal{P}_R$ in the semi-infinite matrix, i.e. $\tilde{\mathcal{P}}_L^{-1} \tilde{\mathcal{P}}_R$ and considering identity (3.37) and identity (3.38), eq.(5.17) can be written as following

$$(5.18) \quad (\tilde{\mathcal{P}}_L^{-1} \tilde{\mathcal{P}}_R)(t) = \begin{pmatrix} 1 & P_1(t_\alpha) & P_2(t_\alpha) & \dots \\ 0 & 1 & P_1(t_\alpha) & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} (\tilde{\mathcal{P}}_L^{-1} \tilde{\mathcal{P}}_R(0)) \begin{pmatrix} 1 & 0 & 0 & \dots \\ P_1(t_\beta) & 1 & 0 & \dots \\ P_2(t_\beta) & P_1(t_\beta) & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where P_k are Schur polynomial as defined in (2.10). To construct tau functions, we define the *moment* matrix $M_\infty(t)$ as,

$$M_\infty(t) := (\tilde{\mathcal{P}}_L^{-1} \tilde{\mathcal{P}}_R)(t).$$

Proposition 5.2. *The constrained condition of BTH is equivalent to matrix M_∞ satisfies identity*

$$(5.19) \quad \Lambda^{Nn} M_\infty = M_\infty \Lambda^{-Mn}.$$

Proof. Using constraint on \mathcal{L} , i.e. $\partial_{t_{1,n}}\mathcal{L} = \partial_{t_{0,n}}\mathcal{L}$, we can get $\partial_{t_{1,n}}M_\infty = \partial_{t_{0,n}}M_\infty$ which can lead to eq.(5.19). \square

Direct calculation can lead to following proposition.

Proposition 5.3. *The matrix M_∞ satisfies identity*

$$\begin{aligned}\partial_{t_{\alpha,n}}M_\infty &= \Lambda^{N(n+1-\frac{\alpha-1}{N})}M_\infty \\ \partial_{t_{\beta,n}}M_\infty &= M_\infty\Lambda^{-M(n+1+\frac{\beta}{M})}.\end{aligned}$$

Let M_i be the $i \times i$ submatrix of M_∞ of the top left corner. By eq.(5.18), each τ -function can be obtained by the determinant [12],

$$\tau_i(t) = \det(M_i(t)).$$

Now, we will consider the detailed structure of tau functions of BTH from the point of reduction of the two-dimensional Toda hierarchy.

As we all know, the tau functions of the two-dimensional Toda lattice hierarchy are given by

$$(5.20) \quad \tau_i = \begin{vmatrix} \bar{C}_{0,0} & \bar{C}_{0,1} & \cdots & \bar{C}_{0,i-1} \\ \bar{C}_{1,0} & \bar{C}_{1,1} & \cdots & \bar{C}_{1,i-1} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{C}_{i-1,0} & \bar{C}_{i-1,1} & \cdots & \bar{C}_{i-1,i-1} \end{vmatrix},$$

where

$$\begin{aligned}\bar{C}_{i,j} &= \int \int \rho(\lambda, \mu) \lambda^i \mu^j e^{\sum_{n=0}^{\infty} x_n \lambda^n + \sum_{n=0}^{\infty} y_n \mu^n} d\lambda d\mu \\ &= \sum_{k,l=0}^{\infty} \bar{c}_{i,j,k,l} P_k(x) P_l(y).\end{aligned}$$

We should note here that the coefficients $\bar{c}_{i,j,k,l}$ are totally independent.

As the original tridiagonal Toda lattice is $(1, 1)$ reduction of the two-dimensional Toda lattice hierarchy. Therefore to get the solution of tridiagonal Toda lattice, we need to add factor $\delta(\lambda - \mu)$ under the integral in the definition of $\bar{C}_{i,j}$, i.e.

$$(5.21) \quad \int \int \rho(\lambda, \mu) \delta(\lambda - \mu) \lambda^i \mu^j e^{\sum_{n=0}^{\infty} x_n \lambda^n + \sum_{n=0}^{\infty} y_n \mu^n} d\lambda d\mu,$$

which can further lead to

$$(5.22) \quad \int \rho(\lambda, \lambda) \lambda^{i+j} e^{\sum_{n=0}^{\infty} (x_n + y_n) \lambda^n} d\lambda.$$

After changing x, y time variables to t_α, t_β , eq.(5.22) become a new function

$$\int \rho(\lambda, \lambda) \lambda^{i+j} e^{\xi_L(\lambda, t_\alpha) + \xi_R(\lambda, t_\beta)} d\lambda$$

which corresponds to $(1, 1)$ -BTH.

Denote ω_N and ω_M as the N -th root and M -th root of unit. For (N, M) -BTH, new function $C_{i,j}$ (new form of $\bar{C}_{i,j}$) have the following form

$$C_{i,j} = \int \int \rho(\lambda, \mu) \delta(\lambda^N - \mu^M) \lambda^i \mu^j e^{\xi_L(\lambda, t_\alpha) + \xi_R(\mu, t_\beta)} d\lambda d\mu$$

$$= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \int \rho(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}}) (\omega_N^p \lambda^{\frac{1}{N}})^i (\omega_M^q \lambda^{\frac{1}{M}})^j e^{\xi_L(\lambda^{\frac{1}{N}}, t_\alpha^p) + \xi_R(\lambda^{\frac{1}{M}}, t_\beta^q)} d\lambda.$$

where

$$(5.23) \quad t_{\alpha,n}^p = (\omega_N^p)^{N(n+1-\frac{\alpha}{N})} t_{\alpha,n}$$

$$(5.24) \quad t_{\beta,n}^q = (\omega_M^q)^{M(n+1+\frac{\beta}{M})} t_{\beta,n}.$$

Transforms (5.23) and (5.24) can be seen as the twist of exponential solutions because of the bigraded structure.

Because in the next we will consider rational solutions of BTH, we write $C_{i,j}$ further into

$$C_{i,j} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{k,l=0}^{\infty} c_{i,j,p,q}^{k,l} P_k(t_\alpha) P_l(t_\beta),$$

where

$$c_{i,j,p,q}^{k,l} = \int \rho(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}}) (\omega_N^p \lambda^{\frac{1}{N}})^i (\omega_M^q \lambda^{\frac{1}{M}})^j (\omega_N^p \lambda^{\frac{1}{N}})^k (\omega_M^q \lambda^{\frac{1}{M}})^l d\lambda.$$

We can find coefficients $\{c_{i,j,p,q}^{k,l}; i, j, k, l \geq 0; 0 \leq p \leq N-1, 0 \leq q \leq M-1\}$ satisfy

$$(5.25) \quad c_{i,j,p,q}^{k+N,l} = c_{i,j,p,q}^{k,l+M},$$

which tells us that $P_m(t_\alpha)P_{n+M}(t_\beta)$ and $P_{m+N}(t_\alpha)P_n(t_\beta)$ always appear at the same time. So the element in position of k row and l column in initial moment matrix can have the following form

$$(5.26) \quad \left(\tilde{\mathcal{P}}_L^{-1} \tilde{\mathcal{P}}_R(0) \right)_{k,l} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} c_{0,0,p,q}^{k-1,l-1}.$$

Therefore tau functions of the BTH can be explicitly written in the form

$$(5.27) \quad \tau_i = \begin{vmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,i-1} \\ C_{1,0} & C_{1,1} & \cdots & C_{1,i-1} \\ \cdots & \cdots & \cdots & \cdots \\ C_{i-1,0} & C_{i-1,1} & \cdots & C_{i-1,i-1} \end{vmatrix}.$$

With the definition of $C_{i,j}$, in the next part we will construct Lax matrix solution using orthogonal polynomials which are nothing but wave function in semi-infinite vector form. Before we do that, we need some formula about property of Schur Hirota derivatives described in the following lemma [18].

Lemma 5.4. *Schur derivatives have following formula*

$$(5.28) \quad P_n(\hat{D}_L) \tau_i \cdot \tau_j = \sum_{k+l=n} P_k(\hat{\partial}_L) \tau_i \times P_l(-\hat{\partial}_L) \tau_j,$$

$$(5.29) \quad P_l(\hat{\partial}_L)[0, 1, 2, \dots, i-1]_L = [0, 1, 2, \dots, i-2, i+l-1]_L,$$

$$(5.30) \quad P_l(-\hat{\partial}_L)[0, 1, 2, \dots, i-1]_L = (-1)^l [0, 1, \dots, i-l-1, i-l+1, \dots, i-1, i]_L,$$

$$(5.31) \quad P_n(\hat{D}_R) \tau_i \cdot \tau_j = \sum_{k+l=n} P_k(\hat{\partial}_R) \tau_i \times P_l(-\hat{\partial}_R) \tau_j,$$

$$(5.32) \quad P_l(\hat{\partial}_R)[0, 1, 2, \dots, i-1]_R = [0, 1, 2, \dots, i-2, i+l-1]_R,$$

$$(5.33) \quad P_l(-\hat{\partial}_R)[0, 1, 2, \dots, i-1]_R = (-1)^l[0, 1, \dots, i-l-1, i-l+1, \dots, i-1, i]_R,$$

where

$$(5.34) \quad [k_0, k_1, k_2, \dots, k_{i-1}]_L = \begin{vmatrix} C_{k_0,0} & C_{k_0,1} & \dots & C_{k_0,i-1} \\ C_{k_1,0} & C_{k_1,1} & \dots & C_{k_1,i-1} \\ \dots & \dots & \dots & \dots \\ C_{k_{i-1},0} & C_{k_{i-1},1} & \dots & C_{k_{i-1},i-1} \end{vmatrix},$$

$$(5.35) \quad [k_0, k_1, k_2, \dots, k_{i-1}]_R = \begin{vmatrix} C_{0,k_0} & C_{0,k_1} & \dots & C_{0,k_{i-1}} \\ C_{1,k_0} & C_{1,k_1} & \dots & C_{1,k_{i-1}} \\ \dots & \dots & \dots & \dots \\ C_{i-1,k_0} & C_{i-1,k_1} & \dots & C_{i-1,k_{i-1}} \end{vmatrix},$$

where P_k are Schur polynomial as defined in (2.10).

In fact, Lemma 5.4 is a special case of abstract general formula of Schur function [18]

$$(5.36) \quad S_Y(\hat{\partial})\tau_\phi = \tau_Y(t).$$

Here $\tau_\phi := [0, 1, 2, \dots, i-1]$ is the standard Wronskian determinant, $Y := (Y_0, Y_1, Y_2, \dots, Y_{i-1})$ and

$$S_Y = \begin{vmatrix} P_{Y_{i-1}+i-1} & P_{Y_{i-2}+i-2} & \dots & P_{Y_1+1} & P_{Y_0} \\ P_{Y_{i-1}+i-2} & P_{Y_{i-2}+i-3} & \dots & P_{Y_1} & P_{Y_0-1} \\ \dots & \dots & \dots & \dots & \dots \\ P_{Y_{i-1}+1} & P_{Y_{i-2}} & \dots & P_{Y_1-i+3} & P_{Y_0-i+2} \\ P_{Y_{i-1}} & P_{Y_{i-2}-1} & \dots & P_{Y_1-i+2} & P_{Y_0-i+1} \end{vmatrix}_{j \times j}.$$

$Y = (Y_0, Y_1, Y_2, \dots, Y_{i-1})$ ($Y_0 > Y_1 > Y_2 > \dots > Y_{i-1}$) corresponds to Young diagram with Y_0 boxes at the first row, Y_1 boxes at the second row and so on. Notice formula [18]

$$(5.37) \quad S_Y(-t) = (-1)^{|Y|} S_{Y'}(t),$$

where Y' is the conjugate Young diagram of Y . Eq.(5.36) together with eq.(5.37) further leads to the lemma above easily.

Then we define wave functions $W_L = (W_{L1}, W_{L2}, \dots)$, $\bar{W}_R = (\bar{W}_{R1}, \bar{W}_{R2}, \dots)^T$ and $\hat{W}_R = (\hat{W}_{R1}, \hat{W}_{R2}, \dots)^T$ with

$$(5.38) \quad W_{Li}(\lambda^{\frac{1}{N}}, t) = \frac{e^{\xi_L(\lambda^{\frac{1}{N}}, t)}}{\tau_{i-1}} \begin{vmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,i-2} & 1 \\ C_{1,0} & C_{1,1} & \dots & C_{1,i-2} & \lambda^{\frac{1}{N}} \\ \dots & \dots & \dots & \dots & \dots \\ C_{i-1,0} & C_{i-1,1} & \dots & C_{i-1,i-2} & \lambda^{\frac{i-1}{N}} \end{vmatrix},$$

$$(5.39) \quad \bar{W}_{Rj}(\lambda^{\frac{1}{M}}, t) = \frac{e^{\xi_R(\lambda^{\frac{1}{M}}, t)}}{\tau_j} \begin{vmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,j-1} \\ C_{1,0} & C_{1,1} & \dots & C_{1,j-1} \\ \dots & \dots & \dots & \dots \\ C_{j-2,0} & C_{j-2,1} & \dots & C_{j-2,j-1} \\ 1 & \lambda^{\frac{1}{M}} & \dots & \lambda^{\frac{j-1}{M}} \end{vmatrix},$$

$$(5.40) \quad \hat{W}_{Rj}(\lambda^{\frac{1}{M}}, t) = \frac{e^{\xi_R(\lambda^{\frac{1}{M}})}}{\tau_{j-1}} \begin{vmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,j-1} \\ C_{1,0} & C_{1,1} & \dots & C_{1,j-1} \\ \dots & \dots & \dots & \dots \\ C_{j-2,0} & C_{j-2,1} & \dots & C_{j-2,j-1} \\ 1 & \lambda^{\frac{1}{M}} & \dots & \lambda^{\frac{j-1}{M}} \end{vmatrix},$$

which satisfy the following orthogonality relation,

$$(5.41) \quad \langle W_{Li}, \bar{W}_{Rj} \rangle = \delta_{i,j}, \quad \langle W_{Li}, \hat{W}_{Rj} \rangle = \delta_{i,j} h_j,$$

where

$$h_j := \frac{\tau_j}{\tau_{j-1}},$$

and inner product \langle, \rangle of functions A and B is defined as

$$\langle A, B \rangle := \int \int \rho(\lambda, \mu) \delta(\lambda^N - \mu^M) A(\lambda, t) B(\mu, t) d\lambda d\mu.$$

Therefore tau functions have another form as

$$\tau_m := \det \left(\langle W_{Li}, \hat{W}_{Rj} \rangle \right)_{1 \leq i, j \leq m}.$$

The entries of the matrix representation of the Lax operator \mathcal{L} can be then calculated by

$$\begin{aligned} a_{ij} &= \frac{P_{i-j+N}(\hat{D}_L) \tau_j \tau_{i-1}}{\tau_{i-1} \tau_j} \\ &= \frac{1}{\tau_{i-1} \tau_j} \sum_{m+n=i-j+N} P_m(\hat{\partial}_L) \tau_j P_n(-\hat{\partial}_L) \tau_{i-1} \\ &= \frac{1}{\tau_{i-1} \tau_j} \sum_{m=0}^{i-j+N} [0, 1, \dots, j-2, j+m-1]_L \\ &\quad (-1)^{m+j-i-N} [0, 1, \dots, j-N+m-2, j-N+m, \dots, i-1]_L \\ &= \left\langle \frac{\lambda e^{\xi_L(\lambda^{\frac{1}{N}}, t)}}{\tau_{i-1}} \begin{vmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,i-2} & 1 \\ C_{1,0} & C_{1,1} & \dots & C_{1,i-2} & \lambda^{\frac{1}{N}} \\ \dots & \dots & \dots & \dots & \dots \\ C_{i-1,0} & C_{i-1,1} & \dots & C_{i-1,i-2} & \lambda^{\frac{i-1}{N}} \end{vmatrix}, \frac{e^{\xi_R(\lambda^{\frac{1}{M}}, t)}}{\tau_j} \begin{vmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,j-1} \\ C_{1,0} & C_{1,1} & \dots & C_{1,j-1} \\ \dots & \dots & \dots & \dots \\ C_{j-2,0} & C_{j-2,1} & \dots & C_{j-2,j-1} \\ 1 & \lambda^{\frac{1}{M}} & \dots & \lambda^{\frac{j-1}{M}} \end{vmatrix} \right\rangle. \end{aligned}$$

So

$$(5.42) \quad a_{i,j} = \langle \lambda W_{Li}, \bar{W}_{Rj} \rangle$$

which are given by the matrix representations of the eigenvalue problems $\mathcal{L}W_L = \lambda W_L$ and $\bar{W}_R \mathcal{L} = \lambda \bar{W}_R$ (see [12] for the details). Till now, we have solved the BTH using orthogonal polynomials.

If we denote \tilde{W}_L and \tilde{W}_R^{-1} as corresponding matrix forms of \mathcal{W}_L (5.15) and \mathcal{W}_R^{-1} (5.16) respectively, then W_L and \bar{W}_R can be represented by matrices \tilde{W}_L and \tilde{W}_R^{-1} respectively as following.

Because

$$\Lambda \begin{pmatrix} 1 \\ \lambda^{\frac{1}{N}} \\ \lambda^{\frac{2}{N}} \\ \vdots \\ \vdots \end{pmatrix} = \lambda^{\frac{1}{N}} \begin{pmatrix} 1 \\ \lambda^{\frac{1}{N}} \\ \lambda^{\frac{2}{N}} \\ \vdots \\ \vdots \end{pmatrix},$$

we can get

$$\begin{aligned} W_L &= \tilde{W}_L \begin{pmatrix} 1 \\ \lambda^{\frac{1}{N}} \\ \lambda^{\frac{2}{N}} \\ \vdots \\ \vdots \end{pmatrix} = \tilde{\mathcal{P}}_L(x, t, \Lambda) \begin{pmatrix} 1 \\ \lambda^{\frac{1}{N}} \\ \lambda^{\frac{2}{N}} \\ \vdots \\ \vdots \end{pmatrix} e^{\xi_L(\lambda^{\frac{1}{N}}, t)} \\ &= \begin{pmatrix} 1 \\ \frac{P_1(-\hat{\partial}_L)\tau_1}{\tau_1} + \lambda^{\frac{1}{N}} \\ \frac{P_2(-\hat{\partial}_L)\tau_2}{\tau_2} + \frac{P_1(-\hat{\partial}_L)\tau_2}{\tau_2} \lambda^{\frac{1}{N}} + \lambda^{\frac{2}{N}} \\ \vdots \\ \vdots \end{pmatrix} e^{\xi_L(\lambda^{\frac{1}{N}}, t)}; \end{aligned}$$

where $\tilde{\mathcal{P}}_L(x, t, \Lambda)$ is as matrix (3.36). This agree with the definition of W_{Li} in eq.(5.38). Also similarly we can get

$$\begin{aligned} \bar{W}_R &= \begin{pmatrix} 1 & \lambda^{\frac{1}{M}} & \lambda^{\frac{2}{M}} & \cdot & \cdot \end{pmatrix} \tilde{W}_R^{-1} = e^{\xi_R(\lambda^{\frac{1}{M}}, t)} \begin{pmatrix} 1 & \lambda^{\frac{1}{M}} & \lambda^{\frac{2}{M}} & \cdot & \cdot \end{pmatrix} \tilde{\mathcal{P}}_R^{-1}(x, t, \Lambda) \\ &= e^{\xi_R(\lambda^{\frac{1}{M}}, t)} \begin{pmatrix} \frac{\tau_0}{\tau_1}, & \frac{P_1(-\hat{\partial}_R)\tau_1}{\tau_2} + \lambda^{\frac{1}{M}} \frac{\tau_1}{\tau_2}, & \frac{P_2(-\hat{\partial}_R)\tau_2}{\tau_3} + \frac{P_1(-\hat{\partial}_R)\tau_2}{\tau_3} \lambda^{\frac{1}{M}} + \frac{\tau_2}{\tau_3} \lambda^{\frac{2}{M}}, & \cdot & \cdot \end{pmatrix}, \end{aligned}$$

where $\tilde{\mathcal{P}}_R^{-1}(x, t, \Lambda)$ is as matrix (3.39). This also agrees with the definition of \bar{W}_{Ri} in eq.(5.39).

Formal factorization mentioned above is about infinite-sized Lax matrix. In the next section, we will consider its finite-sized truncation. Then we can find finite-sized Lax matrix is in fact nilpotent because of dressing structure (3.21). This finite-sized case corresponds to rational solutions of the BTH which will be considered in the next section.

6. RATIONAL SOLUTIONS OF THE (N, M) -BTH

It is well known that the τ -function of the original tridiagonal Toda lattice, i.e. $(1, 1)$ -BTH, has the Schur polynomial solutions associated with rectangular Young diagrams. So what kind of Young diagrams correspond to the BTH become an interesting question. In this section, we only consider homogeneous rational solution of (N, M) -BTH which is one kind of most interesting solutions in nonlinear integrable systems.

In order to describe homogeneous rational solution of (N, M) -BTH ($N \leq M$), we firstly set $\deg(t_{1,0}) = \deg(t_{0,0}) = MN$. Then we get $\deg(P_m(t_\alpha)) = mM$ and $\deg(P_n(t_\beta)) = nN$. Define

$$c_{p,q}^{m,n} := c_{0,0,p,q}^{m,n},$$

and choose the following special homogeneous polynomials $\bar{P}_k(t_\alpha, t_\beta)$ with degree k as τ_1 ,

$$\bar{P}_k(t_\alpha, t_\beta) := \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{mM+nN=k} c_{p,q}^{m,n} P_m(t_\alpha) P_n(t_\beta),$$

where t_α and t_β denote

$$\begin{aligned} t_\alpha &= \{t_{\alpha,n} : 1 \leq \alpha \leq N, n = 0, 1, 2, \dots\} \\ t_\beta &= \{t_{\beta,n} : -M + 1 \leq \beta \leq 0, n = 0, 1, 2, \dots\}, \end{aligned}$$

and k can be any number in the set $\{mM + nN; m, n \in \mathbb{Z}_+\}$. The polynomials $P_m(t_\alpha)$ and $P_n(t_\beta)$ are the elementary Schur polynomials, and they satisfy the following relations,

$$\frac{\partial P_m(t_\alpha)}{\partial t_{\alpha',m'}} = P_{m-N(m'+1)+\alpha'-1}(t_\alpha), \quad \frac{\partial P_n(t_\beta)}{\partial t_{\beta',n'}} = P_{n-M(n'+1)-\beta'}(t_\beta).$$

Note here that $\{c_{p,q}^{m,n} | 0 \leq m, p \leq N-1, 0 \leq n, q \leq M-1\}$ can be arbitrary constants.

Define

$$\bar{P}_k^{l,l'} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{mM+nN=k} c_{p,q}^{m+l,n+l'} P_m(t_\alpha) P_n(t_\beta),$$

and the rational solutions for (N,M)-BTH have the following diagram representation,

$$D_j = \{k - (j-1)N, k - (j-2)N - M, \dots, k - (j-1)M\}, k = 0, N, M, 2N, N+M, 2M, \dots$$

The difference between two adjacent two numbers in D_j is $M - N$.

We denote the tau function corresponding to $D_j(k)$ as τ_{j,D_j} which have the following form

$$\tau_{j,D_j(k)} = S_{D_j(k)} = \begin{vmatrix} \bar{P}_k^{(0,0)} & \bar{P}_{k-M}^{(1,0)} & \cdots & \bar{P}_{k-(j-1)M}^{(j-1,0)} \\ \bar{P}_{k-N}^{(0,1)} & \bar{P}_{k-M-N}^{(1,1)} & \cdots & \bar{P}_{k-(j-1)M-N}^{(j-1,1)} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{k-(j-1)N}^{(0,j-1)} & \bar{P}_{k-M-(j-1)N}^{(1,j-1)} & \cdots & \bar{P}_{k-(j-1)(M+N)}^{(j-1,j-1)} \end{vmatrix}_{j \times j},$$

where $\tau_{j,D_j(k)}$ denotes the j -th tau function($j \times j$ determinant) generated by $\bar{P}_k^{(0,0)}$. The range of rank j depends on the choice of k . This kind of diagram like D_j is not classical Young diagram. It is a kind of generalized diagram which counts the homogeneous degree which comes from the multiplication of two classical Shur functions. We can call this kind of generalized diagram *degree diagram*. Same as Young diagram, the tau functions represented by degree diagram is also Wronskian form, the derivative is about $\partial_{t_{-M+1,0}}$ or $\partial_{t_{N,0}}$. Because the scale of degree in definition before(e.g. $\deg(P_m(t_\alpha)) = mM$) is bigger than common degree of Schur polynomial($\deg(P_m(t)) = m$), the difference of subscript between two adjacent rows(columns) is $N(M)$ not $1(1)$. From this point, it is also different from Hankel determinant.

In the following we firstly only consider the case when N and M are co-prime integers. When they are not co-prime, just divide them by the GCD of them and use the theory of co-prime case in the following. In fact, we can find for fixed value of p, q , all the coefficients of a homogeneous polynomial will be the same because they all satisfy relation (5.25). Then we get

$$\begin{aligned} \bar{P}_k^{l,l'} &= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{mM+nN=k} c_{p,q}^{m+l,n+l'} P_m(t_\alpha) P_n(t_\beta) \\ &= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} c_{p,q}^{m_1+l,n_1+l'} P_{m_1}(t_\alpha) P_{n_1}(t_\beta) + c_{p,q}^{m_2+l,n_2+l'} P_{m_2}(t_\alpha) P_{n_2}(t_\beta) + \dots \\ &\quad + c_{p,q}^{m_{l(k)}+l,n_{l(k)}+l'} P_{m_{l(k)}}(t_\alpha) P_{n_{l(k)}}(t_\beta) \end{aligned}$$

$$\begin{aligned}
&= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} c_{p,q}^{m_1+l, n_1+l'} \sum_{mM+nN=k} P_m(t_\alpha) P_n(t_\beta) \\
&= c_k^{l,l'} \sum_{mM+nN=k} P_m(t_\alpha) P_n(t_\beta),
\end{aligned}$$

where

$$\begin{aligned}
c_{p,q}^{m_1+l, n_1+l'} &= c_{p,q}^{m_2+l, n_2+l'} = \dots = c_{p,q}^{m_{l(k)}+l, n_{l(k)}+l'}, \\
c_k^{l,l'} &= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} c_{p,q}^{m_1+l, n_1+l'},
\end{aligned}$$

$$m_i M + n_i N = k, \quad m_{i+1} = m_i - N, \quad n_{i+1} = n_i + M, \quad 1 \leq i \leq l(k),$$

and number $l(k)$ depends on k . For simplicity, here we just consider the case with all coefficients $c_k^{l,l'}$ equal 1 which is our central consideration in this section and define

$$\bar{P}_k = \sum_{mM+nN=k} P_m(t_\alpha) P_n(t_\beta) = P_{m_1}(t_\alpha) P_{n_1}(t_\beta) + P_{m_2}(t_\alpha) P_{n_2}(t_\beta) + \dots + P_{m_{l(k)}}(t_\alpha) P_{n_{l(k)}}(t_\beta).$$

Then the τ -function τ_s generated by $\tau_1 = \bar{P}_k$ can be expressed by a double-Wronskian determinant,

$$\begin{aligned}
\tau_s(k, t_\alpha, t_\beta) &= \begin{vmatrix} \bar{P}_k & \bar{P}_{k-M} & \dots & \bar{P}_{k-(s-1)M} \\ \bar{P}_{k-N} & \bar{P}_{k-M-N} & \dots & \bar{P}_{k-(s-1)M-N} \\ \dots & \dots & \dots & \dots \\ \bar{P}_{k-(s-1)N} & \bar{P}_{k-M-(s-1)N} & \dots & \bar{P}_{k-(s-1)(M+N)} \end{vmatrix}_{s \times s} \\
&= \begin{vmatrix} P_{m_1}(t_\alpha) & P_{m_2}(t_\alpha) & \dots & P_{m_l}(t_\alpha) \\ P_{m_1-1}(t_\alpha) & P_{m_2-1}(t_\alpha) & \dots & P_{m_l-1}(t_\alpha) \\ P_{m_1-2}(t_\alpha) & P_{m_2-2}(t_\alpha) & \dots & P_{m_l-2}(t_\alpha) \\ \dots & \dots & \dots & \dots \\ P_{m_1-s+1}(t_\alpha) & P_{m_2-s+1}(t_\alpha) & \dots & P_{m_l-s+1}(t_\alpha) \end{vmatrix} \\
&\quad \times \begin{vmatrix} P_{n_1}(t_\beta) & P_{n_1-1}(t_\beta) & P_{n_1-2}(t_\beta) & \dots & P_{n_1-s+1}(t_\beta) \\ P_{n_2}(t_\beta) & P_{n_2-1}(t_\beta) & P_{n_2-2}(t_\beta) & \dots & P_{n_2-s+1}(t_\beta) \\ \dots & \dots & \dots & \dots & \dots \\ P_{n_l}(t_\beta) & P_{n_l-1}(t_\beta) & P_{n_l-2}(t_\beta) & \dots & P_{n_l-s+1}(t_\beta) \end{vmatrix}_{s \times s}.
\end{aligned}$$

Conversely, for a fixed size j of Lax matrix for (N, M) -BTH, the choices of k for τ_1 is in set K_j

$$(6.1) \quad K_j := \{k | k = (j-1)NM + mM + nN, m, n \in \mathbb{Z}_+, 0 \leq m < N, 0 \leq n < M\}.$$

We can see that the number of elements in set K_j is NM (When they are not co-prime, this number will be $\frac{NM}{(N,M)^2}$, where (N, M) is GCD of N and M). When the values of N, M, j, m, n are chosen, a series of non-vanishing tau functions corresponding to them will be fixed.

In the following, we will consider rational solutions of (N, M) -BTH with finite-sized Lax matrix. For (N, M) -BTH, the size of minimal Lax matrix is $(M+1) \times (M+1)$. This minimal Lax matrix has M non-vanishing τ -functions and anyone's degree has $M-N$ jumps between adjacent rows. For the special case of $N = M$, degree diagrams are always rectangle which can also be seen from definition of D_j .

Besides considering the degree diagrams, it is more interesting to consider the decomposition of degree diagrams into representation of Young diagrams. In fact the Young diagram representation of general BTH has a form of multiplication of two different groups of Young diagrams which will be shown in the following theorem.

Theorem 6.1. *For $j \times j$ ($j \geq M+1$)-sized Lax matrix of (N, M) -BTH (denoted as $(N, M)_{j \times j}$), after choosing the value of k as $(j-1)MN + mM + nN$, Young diagram representation of a series of corresponding tau functions are as following*

$$\begin{aligned}
\tau_1 &= \sum_{0 \leq a \leq j-1} S_{((j-1-a)N+m)}(t_\alpha) S_{(n+aM)}(t_\beta), \\
\tau_2 &= \sum_{0 \leq a < b \leq j-1} S_{((j-1-a)N+m-1, (j-1-b)N+m)}(t_\alpha) S_{(n+bM-1, n+aM)}(t_\beta), \\
\tau_3 &= \sum_{0 \leq a < b < c \leq j-1} S_{((j-1-a)N+m-2, (j-1-b)N+m-1, (j-1-c)N+m)}(t_\alpha) S_{(n+cM-2, n+bM-1, n+aM)}(t_\beta), \\
&\dots \dots \dots \\
\tau_s &= \sum_{0 \leq a_1 < a_2 < \dots < a_s \leq j-1} S_{((j-1-a_1)N+m-s+1, (j-1-a_2)N+m-s+2, \dots, (j-1-a_s)N+m)}(t_\alpha) \\
&\quad S_{(n+a_sM-s+1, \dots, n+a_2M-1, n+a_1M)}(t_\beta), \\
&\dots \dots \dots \\
\tau_j &= S_{((j-1)(N-1)+m, (j-2)(N-1)+m, \dots, m)}(t_\alpha) S_{(n+(j-1)(M-1), n+(j-2)(M-1), \dots, n)}(t_\beta).
\end{aligned}$$

Proof. To prove this theorem, one need to use Cauchy-Binet formula. The process is quite complicated because of huge sizes of matrices. So we will omit the proof. One can understand the patten by the following example, i.e. $(2, 3)$ -BTH. \square

To see it clearly, we give some specific examples in the following.

Example 6.2. $(2, 3)$ -BTH, has 6 sets of τ -functions for each size of Lax matrix and the degree diagram for every tau function has one jump between adjacent rows. See $(2, 3)_{4 \times 4}$ in detail as following degree diagram

$$(6.2) \quad (2, 3)_{4 \times 4} \left\{ \begin{array}{l} (0, 0) : \tau_{1, \{18\}} \rightarrow \tau_{2, \{16, 15\}} \rightarrow \tau_{3, \{14, 13, 12\}} \rightarrow \tau_{4, \{12, 11, 10, 9\}}, \\ (0, 1) : \tau_{1, \{20\}} \rightarrow \tau_{2, \{18, 17\}} \rightarrow \tau_{3, \{16, 15, 14\}} \rightarrow \tau_{4, \{14, 13, 12, 11\}}, \\ (1, 0) : \tau_{1, \{21\}} \rightarrow \tau_{2, \{19, 18\}} \rightarrow \tau_{3, \{17, 16, 15\}} \rightarrow \tau_{4, \{15, 14, 13, 12\}}, \\ (0, 2) : \tau_{1, \{22\}} \rightarrow \tau_{2, \{20, 19\}} \rightarrow \tau_{3, \{18, 17, 16\}} \rightarrow \tau_{4, \{16, 15, 14, 13\}}, \\ (1, 1) : \tau_{1, \{23\}} \rightarrow \tau_{2, \{21, 20\}} \rightarrow \tau_{3, \{19, 18, 17\}} \rightarrow \tau_{4, \{17, 16, 15, 14\}}, \\ (1, 2) : \tau_{1, \{25\}} \rightarrow \tau_{2, \{23, 22\}} \rightarrow \tau_{3, \{21, 20, 19\}} \rightarrow \tau_{4, \{19, 18, 17, 16\}}, \end{array} \right.$$

where $\{(p, q), 0 \leq p < 2, 0 \leq q < 3\}$ denote the value of (m, n) in value of k , i.e. (6.1). Here k takes values in $\{18, 20, 21, 22, 23, 25\}$, i.e. the values in bracket of $\tau_{1, \{\cdot\}}$. Every tau function $\tau_{1, \{\cdot\}}$ generates a series of tau functions which are connected by right arrow in (6.2). $\tau_{l, \{\cdot, \dots, \cdot\}}$ represents the l -th tau function whose degree diagram is in the bracket $\{\cdot, \dots, \cdot\}$. In (6.2), we can use product of Young diagrams to represent the four tau functions of the first line, i.e.

(0, 0) case as following by Cauchy-Binet formula

$$\begin{aligned}
\tau_1 &= \tau_{1,\{18\}} = \left| \begin{pmatrix} P_6(t_\alpha) & P_4(t_\alpha) & P_2(t_\alpha) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ P_3(t_\beta) \\ P_6(t_\beta) \\ P_9(t_\beta) \end{pmatrix} \right| \\
&= S_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_\phi(t_\beta) + S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}(t_\beta) + S_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}(t_\beta) + S_\phi(t_\alpha) S_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}}(t_\beta); \\
\tau_2 &= \tau_{2,\{16,15\}} = \left| \begin{pmatrix} P_6(t_\alpha) & P_4(t_\alpha) & P_2(t_\alpha) & 1 \\ P_5(t_\alpha) & P_3(t_\alpha) & P_1(t_\alpha) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ P_3(t_\beta) & P_2(t_\beta) \\ P_6(t_\beta) & P_5(t_\beta) \\ P_9(t_\beta) & P_8(t_\beta) \end{pmatrix} \right| \\
&= S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}(t_\beta) + S_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\beta) + S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}(t_\beta) \\
&\quad + S_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}(t_\beta) + S_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}}(t_\beta); \\
\tau_3 &= \tau_{3,\{14,13,12\}} = \left| \begin{pmatrix} P_6(t_\alpha) & P_4(t_\alpha) & P_2(t_\alpha) & 1 \\ P_5(t_\alpha) & P_3(t_\alpha) & P_1(t_\alpha) & 0 \\ P_4(t_\alpha) & P_2(t_\alpha) & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ P_3(t_\beta) & P_2(t_\beta) & P_1(t_\beta) \\ P_6(t_\beta) & P_5(t_\beta) & P_4(t_\beta) \\ P_9(t_\beta) & P_8(t_\beta) & P_7(t_\beta) \end{pmatrix} \right| \\
&= S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}(t_\beta) + S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}(t_\beta) + S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array}}(t_\beta) \\
&\quad + S_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}}(t_\beta); \\
\tau_4 &= \tau_{4,\{12,11,10,9\}} = \left| \begin{pmatrix} P_6(t_\alpha) & P_4(t_\alpha) & P_2(t_\alpha) & 1 \\ P_5(t_\alpha) & P_3(t_\alpha) & P_1(t_\alpha) & 0 \\ P_4(t_\alpha) & P_2(t_\alpha) & 1 & 0 \\ P_3(t_\alpha) & P_1(t_\alpha) & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ P_3(t_\beta) & P_2(t_\beta) & P_1(t_\beta) & 1 \\ P_6(t_\beta) & P_5(t_\beta) & P_4(t_\beta) & P_3(t_\beta) \\ P_9(t_\beta) & P_8(t_\beta) & P_7(t_\beta) & P_6(t_\beta) \end{pmatrix} \right| \\
&= S_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}(t_\alpha) S_{\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array}}(t_\beta).
\end{aligned}$$

After general theory on homogeneous rational solutions of the (N, M) -BTH, as a special but important case, rational solutions of the $(1, M)$ -BTH will be considered in the next subsection.

6.1. Rational solutions of the $(1, M)$ -BTH. Since the $t_{1,n}$ flows are same as the $t_{0,n}$ flows, we use $t_{1,n} + t_{0,n}$ as a new variable and identify $t_{1,n}$ as $t_{0,n}$. Then the rational solutions for the $(1, M)$ -BTH are obtained from the τ -function,

$$(6.3) \quad \tau_j(k, t) = \left| \begin{array}{cccc} P_k & P_{k-M} & \cdots & P_{k-(j-1)M} \\ P_{k-1} & P_{k-M-1} & \cdots & P_{k-(j-1)M-1} \\ \cdots & \cdots & \cdots & \cdots \\ P_{k-(j-1)} & P_{k-M-(j-1)} & \cdots & P_{k-(j-1)(M+1)} \end{array} \right|_{j \times j},$$

where $P_n = P_n(t_\beta)$ with $t_{0,n} \equiv t_{0,n} + t_{1,n}$. This τ -function can be given by the Schur polynomial associated with the Young diagram which is same as degree diagram mentioned above for $(1, M)$ -BTH, i.e. $\tau_j(k) = S_{Y_j}(k)$ with

$$Y_j(k) = (k-j+1, k-j+1-(M-1), \dots, k-(j-1)M) \quad \text{for } j = 1, 2, \dots, 1 + \left\lfloor \frac{k}{M} \right\rfloor,$$

where $\lfloor \frac{k}{M} \rfloor$ denotes the biggest integer which is less than or equal $\frac{k}{M}$. Note here that the number of boxes in the diagram increases by $M - 1$ between adjacent rows. Let us now give some examples of the τ -functions for the $(1, M)$ -BTH with specific size r of the Lax matrix, denoted by $(1, M)_{r \times r}$. For a given size $r(> M)$, the choices of k are in the set $\{(r-1)M, \dots, rM-1\}$, that is, there are M choices of the first member of the τ -functions, $\tau_1 = P_k$. Then each value of k generates r tau functions ordered from τ_1 to τ_r .

Example 6.3. $(1, 1)$ -BTH, i.e. the original Toda lattice, has only one set of τ -functions for each size of Lax matrix;

$$(6.4) \quad \begin{cases} (1, 1)_{2 \times 2} & : \quad \tau_{\square} \rightarrow \tau_{\phi}, \\ (1, 1)_{3 \times 3} & : \quad \tau_{\square\square} \rightarrow \tau_{\begin{smallmatrix} \square & \\ \square & \end{smallmatrix}} \rightarrow \tau_{\phi}, \\ (1, 1)_{4 \times 4} & : \quad \tau_{\square\square\square} \rightarrow \tau_{\begin{smallmatrix} \square & & \\ \square & & \\ \square & & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \\ \square & \square \end{smallmatrix}} \rightarrow \tau_{\phi}, \\ \dots & : \quad \dots, \end{cases}$$

where $\tau_{\phi} = 1$.

$(1, 2)$ -BTH has two sets of τ -functions, and the Young diagram for each τ -function has one jump between adjacent rows:

$$(6.5) \quad \begin{cases} (1, 2)_{3 \times 3} & : \quad \tau_{\square\square\square} \rightarrow \tau_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \\ \square & \square \end{smallmatrix}}, \\ & \tau_{\begin{smallmatrix} \square & \square & \square \\ \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \\ \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \\ \square & \square & \square \end{smallmatrix}}, \\ (1, 2)_{4 \times 4} & : \quad \tau_{\square\square\square\square} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \\ \square & \square & \square \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \\ \square & \square & \square \end{smallmatrix}}, \\ & \tau_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \square \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \square \end{smallmatrix}}, \\ \dots & : \quad \dots \end{cases}$$

Similarly $(1, 3)$ -BTH has three sets of tau functions, and the Young diagram has two jumps between adjacent rows.

$$\begin{cases} (1, 3)_{4 \times 4} & : \quad \tau_{\square\square\square\square} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \\ \square & \square & \square \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \\ \square & \square & \square \end{smallmatrix}}, \\ & \tau_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \square \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \square \end{smallmatrix}}, \\ & \tau_{\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \square & \\ \square & \square & \square & \square & \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \square & \\ \square & \square & \square & \square & \square \end{smallmatrix}} \rightarrow \tau_{\begin{smallmatrix} \square & \square & \square & \square & \\ \square & \square & \square & \square & \square \end{smallmatrix}}, \\ \dots & : \quad \dots \end{cases}$$

7. CONCLUSIONS AND DISCUSSIONS

We proved the equivalence between (N, M) -BTH and (M, N) -BTH, derived the primary Hirota equations of the (N, M) -BTH, and found several explicit formulas about solutions for the BTH using orthogonal polynomials in the matrix form. We also constructed some rational solutions of the BTH which are parameterized by the products of Schur polynomials corresponding to non-rectangular Young diagrams. It may be interesting to find their significance in terms of the representation theory, as in the case of the original Toda lattice where the rational solutions are given by the Schur polynomials of rectangular Young diagrams and they are the Virasoro singular vectors.

Acknowledgments: This work was carried out under the guidance of Professor Yuji Kodama during my visit to Ohio State University. I would like to thank Professor Yuji Kodama for his guidance

and many useful discussions. I would also like to thank Department of Mathematics at Ohio State University for providing me a generous support and making my visit so pleasant. I also thank Professor Jingsong He(NBU, China) for useful discussions and his general support.

REFERENCES

- [1] G. Carlet, The extended bigraded Toda hierarchy, J. Phys. A, 39(2006), 9411-9435, arXiv:math-ph/0604024.
- [2] C. Z. Li, J. S. He, K. Wu, Y. Cheng, Tau function and Hirota bilinear equations for the extended bigraded Toda hierarchy, Journal of Mathematical Physics, 51(2010), 043514, arXiv:0906.0624.
- [3] M. Toda, Vibration of a chain with nonlinear interaction. J. Phys. Soc. Jpn. 22(1967), 431-436.
- [4] M. Toda, Nonlinear waves and solitons(Kluwer Academic Publishers, Dordrecht, Holland, 1989).
- [5] K. Takasaki, Two extensions of 1-D Toda hierarchy, J. Phys. A: Math. Theor. 43(2010), 434032, arXiv:1002.4688.
- [6] K. Ueno, K. Takasaki, Toda lattice hierarchy, In “*Group representations and systems of differential equations*” (Tokyo, 1982), 1-95, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam, 1984.
- [7] C. Z. Li, J. S. He, Y. C. Su, Block type symmetry of bigraded Toda hierarchy, in preparation.
- [8] T. Milanov, H. H. Tseng, The spaces of Laurent polynomials, \mathbb{P}^1 -orbifolds, and integrable hierarchies, Journal für die reine und angewandte Mathematik, 622 (2008), 189-235, (also see, arXiv:math.AG/0607012).
- [9] M. Blaszak, A. Szum, Lie algebraic approach to the construction of $(2+1)$ -dimensional lattice-field and field integrable Hamiltonian equations, J. Math. Phys. 42(2001), 225-259.
- [10] M. Adler, P. V. Moerbeke, Darboux transforms on band matrices, weights, and associated polynomials, International Mathematics Research Notices, No.18(2001).
- [11] Y. Kodama, V. U. Pierce, Combinatorics of dispersionless integrable systems and universality in random matrix theory. Commun. Math. Phys. 292(2009), arXiv:0811.0351.
- [12] Y. Kodama, J. Ye, Iso-Spectral deformations of general matrix and their reductions on Lie algebras, Commun. Math. Phys. 178(1996), 765-788.
- [13] M. Adler, P. van Moerbeke, Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems, Commun. Math. Phys. 207(1999), 589-620.
- [14] M. Blaszak, K. Marciniak. *R*-matrix approach to lattice integrable systems. J. Math. Phys. 35:9(1994), 4661-4682.
- [15] A. K. Svinin. A class of integrable lattices and KP hierarchy. J. Phys. A: Math. Gen. 34(2001), 10559-10568.
- [16] Y. T. Wu, X. B. Hu. A new integrable differential-difference system and its explicit solutions. J. Phys. A: Math. Gen. 32(1999), 1515-1521.
- [17] G. F. Yu, C. X. Li, J. X. Zhao, X. B. Hu, On a special two-dimensional lattice by Blaszak and Szum: pfaffianization and molecule solutions, Journal of Nonlinear Mathematical Physics, 12, 2(2005), 316-332.
- [18] Y. Ohta, J. Satsuma, D. Takahashi, T. Tokihiro, An elementary introduction to Sato theory, Prog. Theor. Phys. Suppl. 94(1988), 210-241.